SOME ASPECTS OF SENSITIVITY IN GAUSSIAN BAYESIAN NETWORKS

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Def. A ***Bayesian network*** is a pair $(\mathcal{G},\mathcal{P})$ where $\mathcal{G}$ is a directed acyclic graph (DAG), with nodes representing random variables $\boldsymbol{X}=\{X_1,\ldots,X_n\}$ and edges representing probabilistic dependences, and $\mathcal{P}=\{p(x_1|pa(x_1)),\ldots, p(x_n|pa(x_n))\}$ is a set of conditional probability distributions (one for each variable) where $pa(x_i)$ is the set of parents of node X_i in $\mathcal{G}$.

The set $\mathcal{P}$ defines the associated joint probability density as

$$p(x)=\prod_{i=1}^{n} p(x_i\,|\,pa(x_i))$$

When we have information about the state of a variable, the ***evidence propagation*** updates the probability distribution of the rest of variables with this information or evidence.

In Gaussian Bayesian networks we perform the evidence propagation computing the conditional probability density of a multivariate normal distribution given the set of evidential variable **E**=**e**. Therefore, considering the partition **X**=(**Y**,**E**) being **Y** the set of non-evidential variables and **E** the set of evidential variable, the conditional probability distribution of **Y**, given the evidence **E**, is a multivariate normal distribution.

Def. A ***Gaussian Bayesian network*** is a Bayesian network over $\boldsymbol{X}=\{X_1,\ldots,X_n\}$ where the joint probability density is a multivariate normal distribution $N(\boldsymbol{\mu},\boldsymbol{\Sigma})$, with $\boldsymbol{\mu}$ the n-dimensional mean vector and $\boldsymbol{\Sigma}$ the n x n covariance matrix, with the dependence structure of $\mathcal{G}$.

SENSITIVITY ANALYSIS

· This sensitivity analysis is based on comparing two different models:

- The *original model* $\boldsymbol{X}\sim N(\boldsymbol{\mu},\boldsymbol{\Sigma})$
- The *perturbed model* $\boldsymbol{X}\sim N(\boldsymbol{\mu}^{\delta},\boldsymbol{\Sigma}^{\Delta})$, obtained after adding a *perturbed mean vector* $\boldsymbol{\delta}$ or a *perturbed covariance matrix* $\boldsymbol{\Delta}$ to the inaccurate elements of the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, respectively.

· With these two models the evidence propagation is performed, obtaining the network's output as the conditional distribution of the set of variables of interest **Y** given **E**.

· Both network's output are compared computing the **Kullback-Leibler divergence**.

KULLBACK-LEIBLER DIVERGENCE

To compare the network's output obtained for the original model and for the perturbed model, this expression for the Kullback-Leibler (KL) divergence is used

$$KL^{\mathbf{p}_j}(f,f^{\mathbf{p}_j})=E_f\left[\ln\frac{f}{f^{\mathbf{p}_j}}\right]=$$

$$=\frac{1}{2}\left[\ln\frac{\left|\boldsymbol{\Sigma}^{\mathbf{Y|E},\mathbf{p}_j}\right|}{\left|\boldsymbol{\Sigma}^{\mathbf{Y|E}}\right|}+tr\left(\boldsymbol{\Sigma}^{\mathbf{Y|E}}\left(\boldsymbol{\Sigma}^{\mathbf{Y|E},\mathbf{p}_j}\right)^{-1}\right)+\left(\boldsymbol{\mu}^{\mathbf{Y|E},\mathbf{p}_j}-\boldsymbol{\mu}^{\mathbf{Y|E}}\right)^T\left(\boldsymbol{\Sigma}^{\mathbf{Y|E},\mathbf{p}_j}\right)^{-1}\left(\boldsymbol{\mu}^{\mathbf{Y|E},\mathbf{p}_j}-\boldsymbol{\mu}^{\mathbf{Y|E}}\right)-\dim(\mathbf{Y})\right]$$

being f the conditional probability density obtained for the original model after the evidence propagation and $f^{\mathbf{p}j}$ the conditional probability density obtained for the perturbed model, after the evidence propagation.

PERTURBED MODELS

Five different perturbed models are going to be compared with the original model. Those perturbed models are obtained considering the partition of the perturbed mean vector $\boldsymbol{\delta}$ and the perturbed covariance matrix $\boldsymbol{\Delta}$ as

$$\boldsymbol{\delta}=\begin{pmatrix}\boldsymbol{\delta}_{\mathbf{Y}}\\ \boldsymbol{\delta}_{\mathbf{E}}\end{pmatrix}\quad\text{and}\quad\boldsymbol{\Delta}=\begin{pmatrix}\boldsymbol{\Delta}_{\mathbf{YY}} & \boldsymbol{\Delta}_{\mathbf{YE}}\\ \boldsymbol{\Delta}_{\mathbf{EY}} & \boldsymbol{\Delta}_{\mathbf{EE}}\end{pmatrix}$$

Therefore, there can be uncertainty about

-The means of the variables of interest, being the perturbed model $\boldsymbol{X}\sim(\boldsymbol{\mu}^{\delta\mathbf{Y}},\boldsymbol{\Sigma})$, or about the means of the evidential variables, with the perturbed model $\boldsymbol{X}\sim(\boldsymbol{\mu}^{\delta\mathbf{E}},\boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu}^{\delta_{\mathbf{Y}}}=\begin{pmatrix}\boldsymbol{\mu}_{\mathbf{Y}}+\boldsymbol{\delta}_{\mathbf{Y}}\\ \boldsymbol{\mu}_{\mathbf{E}}\end{pmatrix}\qquad\boldsymbol{\mu}^{\delta_{\mathbf{E}}}=\begin{pmatrix}\boldsymbol{\mu}_{\mathbf{Y}}\\ \boldsymbol{\mu}_{\mathbf{E}}+\boldsymbol{\delta}_{\mathbf{E}}\end{pmatrix}$$

-The variances-covariances between the variables of interest, being the perturbed model $\boldsymbol{X}\sim(\boldsymbol{\mu},\boldsymbol{\Sigma}^{\Delta\mathbf{YY}})$, or about the variances-covariances between the evidential variables, where the perturbed model is given by $\boldsymbol{X}\sim(\boldsymbol{\mu},\boldsymbol{\Sigma}^{\Delta\mathbf{EE}})$, or about the covariances between the variables of interest and the evidential variables, being the perturbed model $\boldsymbol{X}\sim(\boldsymbol{\mu},\boldsymbol{\Sigma}^{\Delta\mathbf{YE}})$, where

$$\boldsymbol{\Sigma}^{\Delta_{\mathbf{YY}}}=\begin{pmatrix}\boldsymbol{\Sigma}_{\mathbf{YY}}+\boldsymbol{\Delta}_{\mathbf{YY}} & \boldsymbol{\Sigma}_{\mathbf{YE}}\\ \boldsymbol{\Sigma}_{\mathbf{EY}} & \boldsymbol{\Sigma}_{\mathbf{EE}}\end{pmatrix}\qquad\boldsymbol{\Sigma}^{\Delta_{\mathbf{EE}}}=\begin{pmatrix}\boldsymbol{\Sigma}_{\mathbf{YY}} & \boldsymbol{\Sigma}_{\mathbf{YE}}\\ \boldsymbol{\Sigma}_{\mathbf{EY}} & \boldsymbol{\Sigma}_{\mathbf{EE}}+\boldsymbol{\Delta}_{\mathbf{EE}}\end{pmatrix}\qquad\boldsymbol{\Sigma}^{\Delta_{\mathbf{YE}}}=\begin{pmatrix}\boldsymbol{\Sigma}_{\mathbf{YY}} & \boldsymbol{\Sigma}_{\mathbf{YE}}+\boldsymbol{\Delta}_{\mathbf{YE}}\\ \boldsymbol{\Sigma}_{\mathbf{EY}}+\boldsymbol{\Delta}_{\mathbf{EY}} & \boldsymbol{\Sigma}_{\mathbf{EE}}\end{pmatrix}$$

RESULTS

Prop 1. Let $(\mathcal{G},\mathcal{P})$ be a Gaussian Bayesian network with $\boldsymbol{X}\sim N(\boldsymbol{\mu},\boldsymbol{\Sigma})$, where $\boldsymbol{X}=\{\mathbf{Y},\mathbf{E}\}$, being **Y** the set of variables of interest and **E** the set of evidential variables. For $\boldsymbol{\delta}$ as the perturbed mean vector, the obtained results are

(1)When the perturbation $\boldsymbol{\delta}_{\mathbf{Y}}$ is added to the mean of the variables of interest the KL divergence is given by

$$KL^{\boldsymbol{\mu}_{\mathbf{Y}}}(f,f^{\boldsymbol{\mu}_{\mathbf{Y}}})=\frac{1}{2}\left[\boldsymbol{\delta}_{\mathbf{Y}}^T\left(\boldsymbol{\Sigma}^{\mathbf{Y|E}}\right)^{-1}\boldsymbol{\delta}_{\mathbf{Y}}\right]$$

(2) When the perturbation $\boldsymbol{\delta}_{\mathbf{E}}$ is added to the mean of the variables of interest the KL divergence is given by

$$KL^{\boldsymbol{\mu}_{\mathbf{E}}}(f,f^{\boldsymbol{\mu}_{\mathbf{E}}})=\frac{1}{2}\left[\boldsymbol{\delta}_{\mathbf{E}}^T\left(\boldsymbol{\Sigma}_{\mathbf{YE}}\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\right)^T\left(\boldsymbol{\Sigma}^{\mathbf{Y|E}}\right)^{-1}\left(\boldsymbol{\Sigma}_{\mathbf{YE}}\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\right)\boldsymbol{\delta}_{\mathbf{E}}\right]$$

Prop 2. Let $(\mathcal{G},\mathcal{P})$ be a Gaussian Bayesian network with $\boldsymbol{X}\sim N(\boldsymbol{\mu},\boldsymbol{\Sigma})$, where $\boldsymbol{X}=\{\mathbf{Y},\mathbf{E}\}$, being **Y** the set of variables of interest and **E** the set of evidential variables. For $\boldsymbol{\Delta}=\begin{pmatrix}\boldsymbol{\Delta}_{\mathbf{YY}} & \boldsymbol{\Delta}_{\mathbf{YE}}\\ \boldsymbol{\Delta}_{\mathbf{EY}} & \boldsymbol{\Delta}_{\mathbf{EE}}\end{pmatrix}$ as the perturbed covariance matrix, the obtained results are

(1) When the perturbation $\boldsymbol{\Delta}_{\mathbf{YY}}$ is added to the variances-covariances between variables in **Y**, the KL divergence is

$$KL^{\boldsymbol{\Sigma}_{\mathbf{YY}}}(f,f^{\boldsymbol{\Sigma}_{\mathbf{YY}}})=\frac{1}{2}\left[\ln\frac{\left|\boldsymbol{\Sigma}^{\mathbf{Y|E}}+\boldsymbol{\Delta}_{\mathbf{YY}}\right|}{\left|\boldsymbol{\Sigma}^{\mathbf{Y|E}}\right|}+tr\left(\boldsymbol{\Sigma}^{\mathbf{Y|E}}\left(\boldsymbol{\Sigma}^{\mathbf{Y|E}}+\boldsymbol{\Delta}_{\mathbf{YY}}\right)^{-1}\right)-\dim(\mathbf{Y})\right]$$

(2) When the perturbation $\boldsymbol{\Delta}_{\mathbf{EE}}$ is added to the variances-covariances between variables in **E**, the KL divergence is

$$KL^{\boldsymbol{\Sigma}_{\mathbf{EE}}}(f,f^{\boldsymbol{\Sigma}_{\mathbf{EE}}})=\frac{1}{2}\left[\ln\frac{\left|\boldsymbol{\Sigma}^{\mathbf{Y|E},\Delta_{\mathbf{EE}}}\right|}{\left|\boldsymbol{\Sigma}^{\mathbf{Y|E}}\right|}+tr\left(\boldsymbol{\Sigma}^{\mathbf{Y|E}}\left(\boldsymbol{\Sigma}^{\mathbf{Y|E},\Delta_{\mathbf{EE}}}\right)^{-1}\right)+\left(\boldsymbol{\mu}^{\mathbf{Y|E},\Delta_{\mathbf{EE}}}-\boldsymbol{\mu}^{\mathbf{Y|E}}\right)^T\left(\boldsymbol{\Sigma}^{\mathbf{Y|E},\Delta_{\mathbf{EE}}}\right)^{-1}\left(\boldsymbol{\mu}^{\mathbf{Y|E},\Delta_{\mathbf{EE}}}-\boldsymbol{\mu}^{\mathbf{Y|E}}\right)-\dim(\mathbf{Y})\right]$$

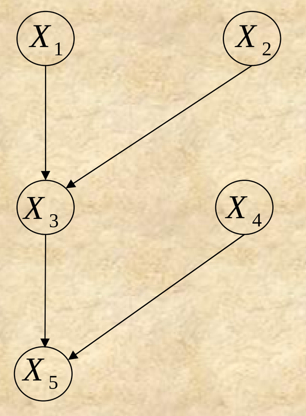
(3) When the perturbation $\boldsymbol{\Delta}_{\mathbf{YE}}$ is added to the covariances between variables in **Y** and variables in **E**, the obtained KL divergence is given by

$$KL^{\boldsymbol{\Sigma}_{\mathbf{YE}}}(f,f^{\boldsymbol{\Sigma}_{\mathbf{YE}}})=\frac{1}{2}\left[\ln\frac{\left|\boldsymbol{\Sigma}^{\mathbf{Y|E}}-M(\boldsymbol{\Delta}_{\mathbf{YE}})\right|}{\left|\boldsymbol{\Sigma}^{\mathbf{Y|E}}\right|}+tr\left(\boldsymbol{\Sigma}^{\mathbf{Y|E}}\left(\boldsymbol{\Sigma}^{\mathbf{Y|E}}-M(\boldsymbol{\Delta}_{\mathbf{YE}})\right)^{-1}\right)+(\mathbf{e}-\boldsymbol{\mu}_{\mathbf{E}})^T\left(\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\right)^T\boldsymbol{\Delta}_{\mathbf{YE}}^T\left(\boldsymbol{\Sigma}^{\mathbf{Y|E}}-M(\boldsymbol{\Delta}_{\mathbf{YE}})\right)^{-1}\boldsymbol{\Delta}_{\mathbf{YE}}\left(\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\right)(\mathbf{e}-\boldsymbol{\mu}_{\mathbf{E}})-\dim(\mathbf{Y})\right]$$

where $M(\boldsymbol{\Delta}_{\mathbf{YE}})=\boldsymbol{\Delta}_{\mathbf{YE}}\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\boldsymbol{\Sigma}_{\mathbf{YE}}^T+\boldsymbol{\Sigma}_{\mathbf{YE}}\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\boldsymbol{\Delta}_{\mathbf{EY}}+\boldsymbol{\Delta}_{\mathbf{YE}}\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\boldsymbol{\Delta}_{\mathbf{EY}}$

EXAMPLE

The Gaussian Bayesian network of the example represents how a machine, made up of 5 elements, works



$$\mathbf{X}=\{X_1,X_2,X_3,X_4,X_5\}\approx N(\mu,\Sigma)$$

$$\boldsymbol{\mu}=\begin{pmatrix}2\\3\\3\\4\\5\end{pmatrix}\qquad\Sigma=\begin{pmatrix}3&0&6&0&6\\0&2&2&0&2\\6&2&15&0&15\\0&0&0&2&4\\6&2&15&4&26\end{pmatrix}$$

$\mathbf{X}=\{\mathbf{E},\mathbf{Y}\}$ where $\mathbf{Y}=\{X_3,X_4,X_5\}$ $\mathbf{E}=\{X_1=2,X_2=4\}$

The perturbed mean vector $\boldsymbol{\delta}$ and the perturbed covariance matrix $\boldsymbol{\Delta}$ are given by

$$\boldsymbol{\delta}=\begin{pmatrix}0\\2\\0\\1\\-1\end{pmatrix}\qquad\boldsymbol{\Delta}=\begin{pmatrix}0&0&-1&0&0\\0&3&0&0&1\\-1&0&2&0&0\\0&0&0&1&2\\0&1&0&2&2\end{pmatrix}$$

Computing the values of the KL divergence, we obtain that *the network is sensitive to the perturbations proposed for the mean vector μ and for $\boldsymbol{\Sigma}_{\mathbf{YE}}$*

$$KL^{\boldsymbol{\mu}_{\mathbf{Y}}}(f,f^{\boldsymbol{\mu}_{\mathbf{Y}}})=1.75$$
$$KL^{\boldsymbol{\mu}_{\mathbf{E}}}(f,f^{\boldsymbol{\mu}_{\mathbf{E}}})=2$$
$$KL^{\boldsymbol{\Sigma}_{\mathbf{YY}}}(f,f^{\boldsymbol{\Sigma}_{\mathbf{YY}}})=0.49$$
$$KL^{\boldsymbol{\Sigma}_{\mathbf{EE}}}(f,f^{\boldsymbol{\Sigma}_{\mathbf{EE}}})=0.203$$
$$KL^{\boldsymbol{\Sigma}_{\mathbf{YE}}}(f,f^{\boldsymbol{\Sigma}_{\mathbf{YE}}})=1.888$$