ANALYZING THE EFFECT OF INTRODUCING A KURTOSIS PARAMETER IN GAUSSIAN BAYESIAN NETWORKS

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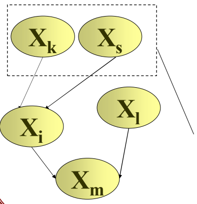
GAUSSIAN BAYESIAN NETWORKS

X = ($X_1, \ldots, X_n$) Multivariate Normal $N_n(\mu,\Sigma)$



$$f(\vec{x}) = \prod_{i=1}^{n} f(x_i \mid pa(x_i))$$

$pa(x_i)$ ⟶ parents of node X_i

EXPONENTIAL POWER DISTRIBUTION

X = ($X_1, \ldots, X_n$) random vector $EP_n(\mu,\Sigma,\beta)$

$$f_\beta(\vec{x}) = \frac{n\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}\Gamma\left(1+\frac{n}{2\beta}\right)2^{1+\frac{n}{2\beta}}} |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left[(\vec{x}-\vec{\mu})^T \Sigma^{-1}(\vec{x}-\vec{\mu})\right]^\beta\right\}$$

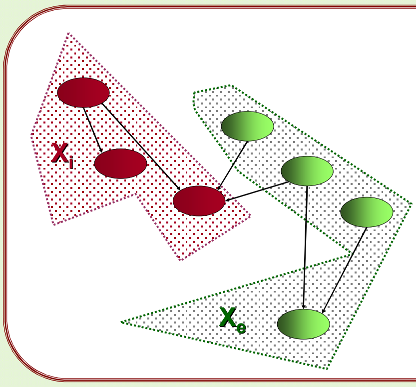
$$\beta \in (0,\infty)$$

$\beta = 1$ Multivariate Normal

$\beta = 0.5$ Multivariate Double Exponential

$\beta \to \infty$ Uniform

NETWORK'S OUTPUT



$$\mathbf{X} = \begin{pmatrix} X_i \\ X_e \end{pmatrix} \begin{matrix} \longrightarrow \text{"interest"} \\ \longrightarrow \text{"evidence"} \end{matrix}$$

Conditional Distribution

f (X_i | X_e)

KULLBACK-LEIBLER DIVERGENCE

A quantity which measures the difference between two probability distributions

$$D_{KL}(p,q) = \int p(x) \log\left(\frac{p(x)}{q(x)}\right) dx$$

SENSITIVITY ANALYSIS

$$\mathbf{X} = \begin{pmatrix} (X_i)^{(p)} \\ (X_e)^{(n-p)} \end{pmatrix} \begin{matrix} \nearrow EP_n(\mu,\Sigma,\beta) \longrightarrow f_\beta(X_i \mid X_e) \sim E_p(\mu_{i.e},\Sigma_{ii.e},g_{i.e}) \text{ (elliptical distribution)} \\ \searrow N_n(\mu,\Sigma) \longrightarrow f(X_i \mid X_e) \sim N_p(\mu_{i.e},\Sigma_{ii.e}) \end{matrix}$$

$\mu_{i.e} = \mu_i + \Sigma_{ie}(\Sigma_{ee})^{-1}(x_e-\mu_e)$; $\Sigma_{ii.e} = \Sigma_{ii} - \Sigma_{ie}(\Sigma_{ee})^{-1}\Sigma_{ei}$

$g_{i.e}(t) = \exp\{-½(t+q_e)^\beta\}$

$q_e = (x_e-\mu_e)^T(\Sigma_{ee})^{-1}(x_e-\mu_e)$

D_{KL} (f (X_i | X_e), f_β (X_i | X_e))

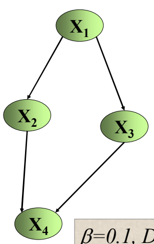
$$f_\beta(x_i|x_e) = \frac{\Gamma\left(\frac{p}{2}\right)}{\pi^{\frac{p}{2}}\int_0^\infty t^{\frac{p}{2}-1}\exp\left\{-\frac{1}{2}(t+q_e)^\beta\right\}dt} |\Sigma_{ii.e}|^{-\frac{1}{2}} \exp-\frac{1}{2}\left\{\left[(x_i-\mu_{i.e})^T(\Sigma_{ii.e})^{-1}(x_i-\mu_{i.e})+q_e\right]^\beta\right\} \quad ; \quad f(x_i|x_e) = \frac{1}{(2\pi)^{\frac{p}{2}}}|\Sigma_{ii.e}|^{-\frac{1}{2}} \exp-\frac{1}{2}\left\{(x_i-\mu_{i.e})^T(\Sigma_{ii.e})^{-1}(x_i-\mu_{i.e})\right\}$$

$$D_{KL}\left(f(x_i|x_e), f_\beta(x_i|x_e)\right) = \int\left[\log\frac{f(x_i|x_e)}{f_\beta(x_i|x_e)}\right] f(x_i|x_e)\,dx_i =$$

$$= \log\frac{\int_0^\infty t^{\frac{p}{2}-1}\exp\left\{-\frac{1}{2}(t+q_e)^\beta\right\}dt}{2^{\frac{p}{2}}\Gamma\left(\frac{p}{2}\right)} - \frac{1}{2}\int\left\{(x_i-\mu_{i.e})^T(\Sigma_{ii.e})^{-1}(x_i-\mu_{i.e}) - \left[(x_i-\mu_{i.e})^T(\Sigma_{ii.e})^{-1}(x_i-\mu_{i.e})+q_e\right]^\beta\right\} f(x_i|x_e)\,dx_i =$$

$$= \log\frac{\int_0^\infty t^{\frac{p}{2}-1}\exp\left\{-\frac{1}{2}(t+q_e)^\beta\right\}dt}{2^{\frac{p}{2}}\Gamma\left(\frac{p}{2}\right)} - \frac{p}{2} + \frac{(q_e)^{\beta+\frac{p}{2}}}{2^{\frac{p}{2}}} \underbrace{U\left(a=\frac{p}{2}, b=\beta+\frac{p}{2}+1, z=\frac{q_e}{2}\right)}_{\text{KUMMER FUNCTION}}$$

EXAMPLE 1



$$\mu = \begin{pmatrix} 3 \\ 4 \\ 9 \\ 14 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 4 & 4 & 8 & 12 \\ 4 & 5 & 8 & 13 \\ 8 & 8 & 20 & 28 \\ 12 & 13 & 28 & 42 \end{pmatrix}$$

"evidence" { X_1=7, X_2=8, X_3=17 } → q_e= 4

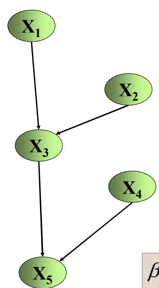
"interest" { X_4 }

β=0.1, D_{KL}=52.732 ; β=0.6, D_{KL}=1.808; β=0.6, D_{KL}=0.418

β=0.9, D_{KL}=0.1043; β=1, D_{KL}=0 ; β=1.1, D_{KL}=0.10.93

β=1.4, D_{KL}=2.046 ; β=2, D_{KL}=21.684; β=3, D_{KL}=326.2

EXAMPLE 2



$$\mu = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 4 \\ 5 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 3 & 0 & 6 & 0 & 6 \\ 0 & 2 & 2 & 0 & 2 \\ 6 & 2 & 15 & 0 & 15 \\ 0 & 0 & 0 & 2 & 4 \\ 6 & 2 & 15 & 4 & 26 \end{pmatrix}$$

"evidence" { X_1=7, X_2=8} → q_e= 20.83

"interest" {X_3, X_4, X_5}

β=0.1, D_{KL}=70.59 ; β=0.6, D_{KL}=2.986; β=0.8, D_{KL}=0.795

β=0.9, D_{KL}=0.213; β=1, D_{KL}=0; β=1.1, D_{KL}=0.259

β=1.4, D_{KL}=6.114 ; β=2, D_{KL}=109.82; β=3, D_{KL}=4484.8