Imperial College London

Application of Global Sensitivity Indices for measuring the effectiveness of Quasi-Monte Carlo methods and parameter estimation

S. Kucherenko

Email: s.kucherenko@ic.ac.uk

Imperial College London

Outline

Advantages and disadvantages of Monte Carlo method

Why Sobol' sequences are so effective ?

Is this true that for high-dimensional problems Quasi Monte Carlo methods offer no practical advantage over Monte Carlo"?

Effective dimension versus nominal dimension and their link with global Sensitivity Indices

Classification of functions and functionals based on global sensitivity indices

Application of parametric GSA for optimal experimental design

Imperial College London

Comparison deterministic and Monte Carlo integration methods

$$I[f] = \int_{H^n} f(\vec{x}) d\vec{x}$$

Deterministic integration method of p-order,

k points in each direction: $N = k^n$

Error: $\varepsilon = O(k^{-p})$, $N = O(1/\varepsilon)^{n/p}$.

Estimate: $\varepsilon = 10^{-2}, p = 2, n = 50 \rightarrow$

$N_d = 10^{50} \approx$ the total number of particles in the universe

$\rightarrow I[f]$ is impossible to evaluate !

"Curse of Dimensionality"

Imperial College London

Monte Carlo integration methods

$$I[f] = E[f(\vec{x})]$$

$$\text{Monte Carlo}: I_N[f] = \frac{1}{N}\sum_{i=1}^{N} f(\vec{z}_i)$$

$\{\vec{z}_i\}$ – is a sequence of random points in H^n

Error: $\varepsilon = \left| I[f] - I_N[f] \right|$

Expectation: $E(\varepsilon^2) = \dfrac{\sigma^2(f)}{N}$

$$\varepsilon_N = (E(\varepsilon^2))^{1/2} = \frac{\sigma(f)}{N^{1/2}} \rightarrow$$

Convergence does not depent on dimensionality

but it is slow

How to improve MC ?

Slow convergence: $\varepsilon_N = \dfrac{\sigma(f)}{N^{1/2}}$

How to improve MC ?

I. Decrease $\sigma(f)$ – variance reduction.

II. Use better (more uniformly distributed) sequences.

Discrepancy is a measure of deviation from uniformity:

$Q(\vec{y}) \in H^{\mathrm{n}},\ Q(\vec{y}) = [0, y_1) \times [0, y_2) \times \ldots \times [0, y_n),$

$m(Q)$ – volume of Q

$$D_N = \sup_{Q(\vec{y}) \in H^n} \left| \frac{N_{Q(\vec{y})}}{N} - m(Q) \right|,$$

random sequences: $D_N \to (\ln\ \ln\ N)/N^{1/2} \sim 1/N^{1/2}$

Quasi random sequences

$$D_N \le c(d)\frac{(\ln N)^n}{N} - \text{Low discrepancy sequences (LDS)}$$

Convergence: $\varepsilon_{QMC} = \left| I[f] - I_N[f] \right| \le V(f) D_N$

$V(f)$ – Variation of f

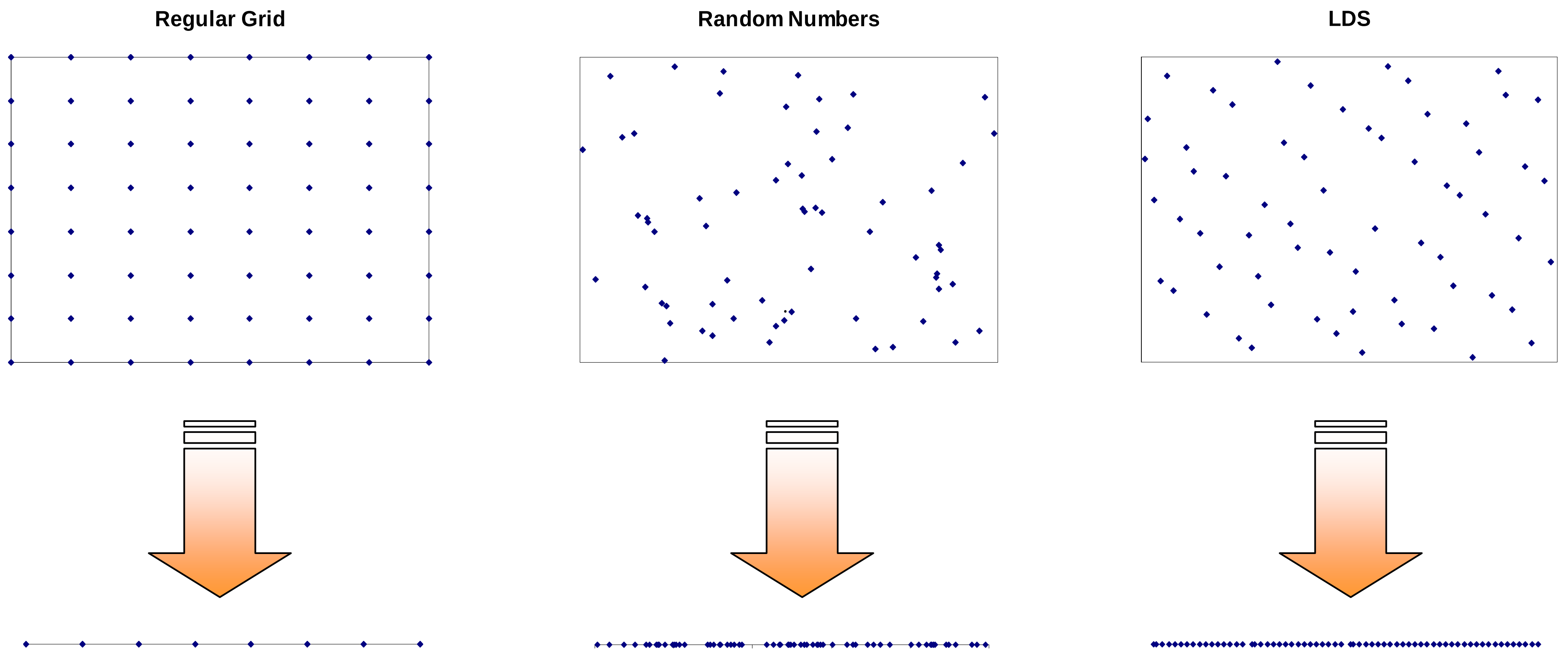
$$\varepsilon_{QMC} = \frac{O(\ln N)^n}{N}$$

Assymptotically $\varepsilon_{QMC} \sim O(1/N) \rightarrow$ much higher than

$\varepsilon_{MC} \sim O(1/\sqrt{N})$

Imperial College London

Projections of Different 2D sequences to 1D

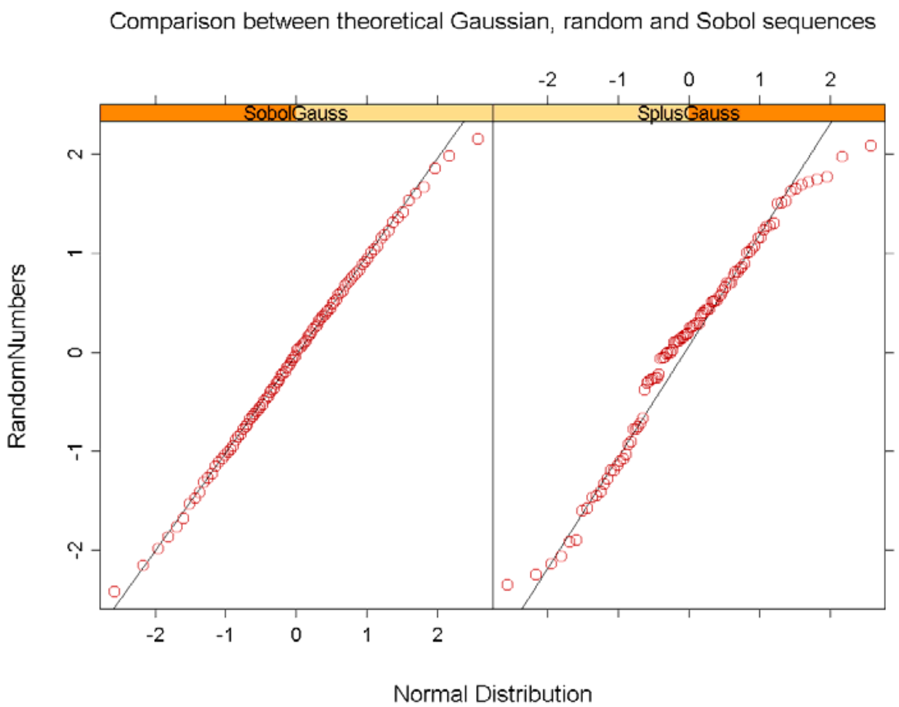


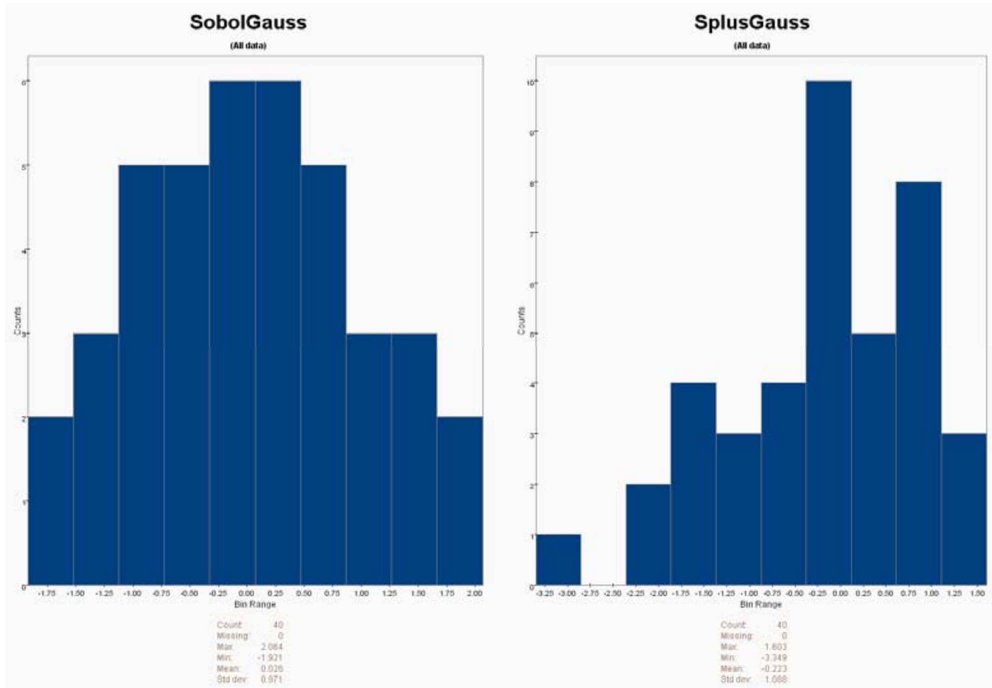
Imperial College London

Normally distributed Sobol' Sequences

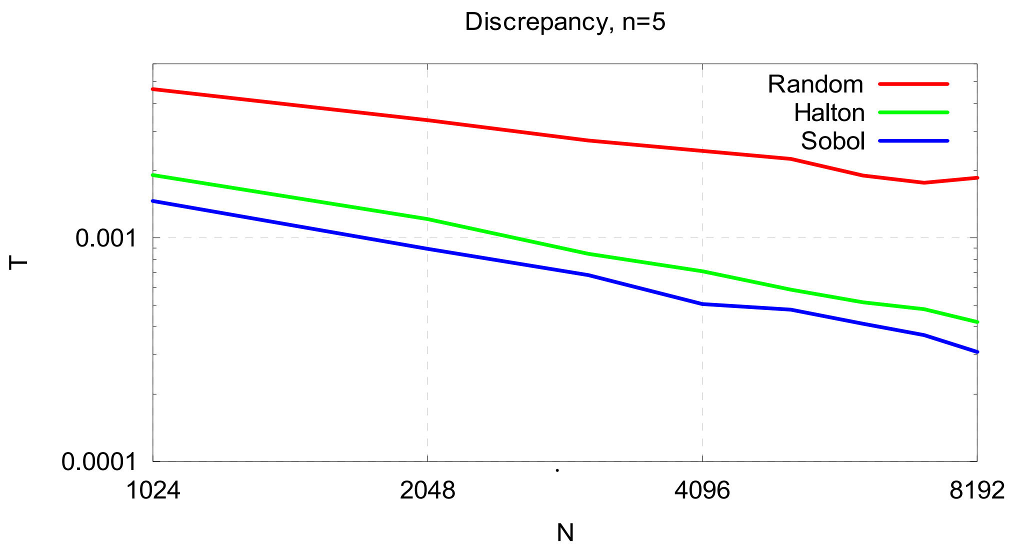
Normal probability plots

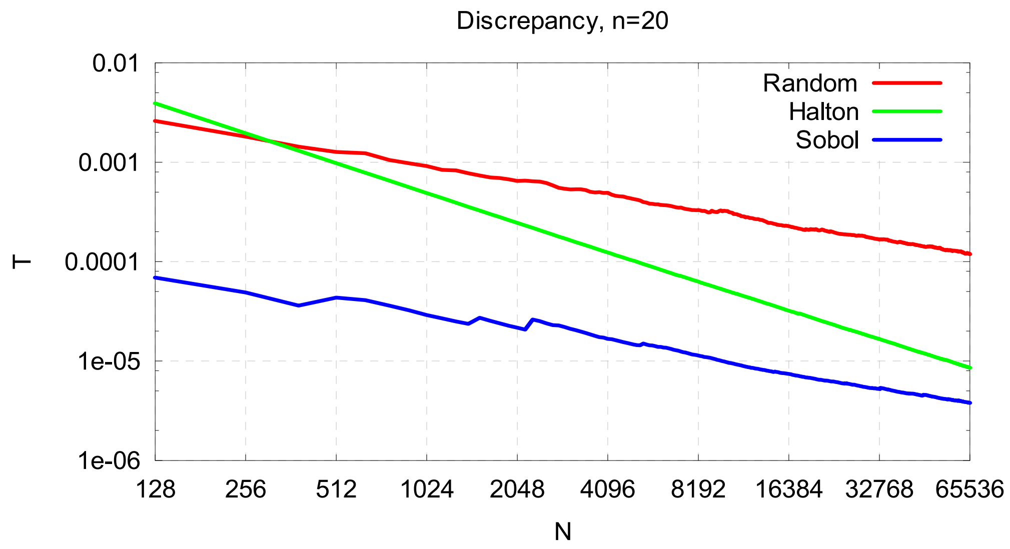
Histograms





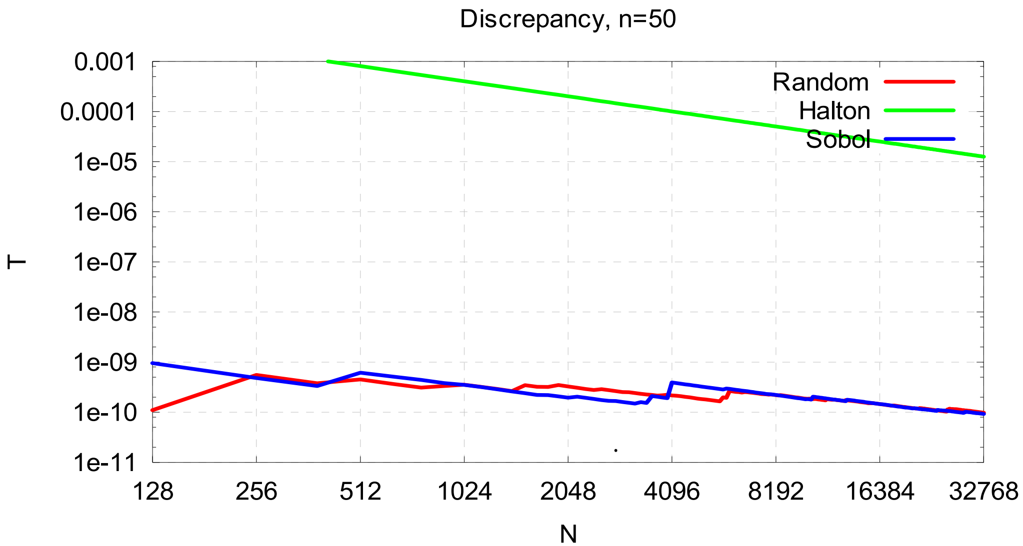
Discrepancy
I. Low Dimensions

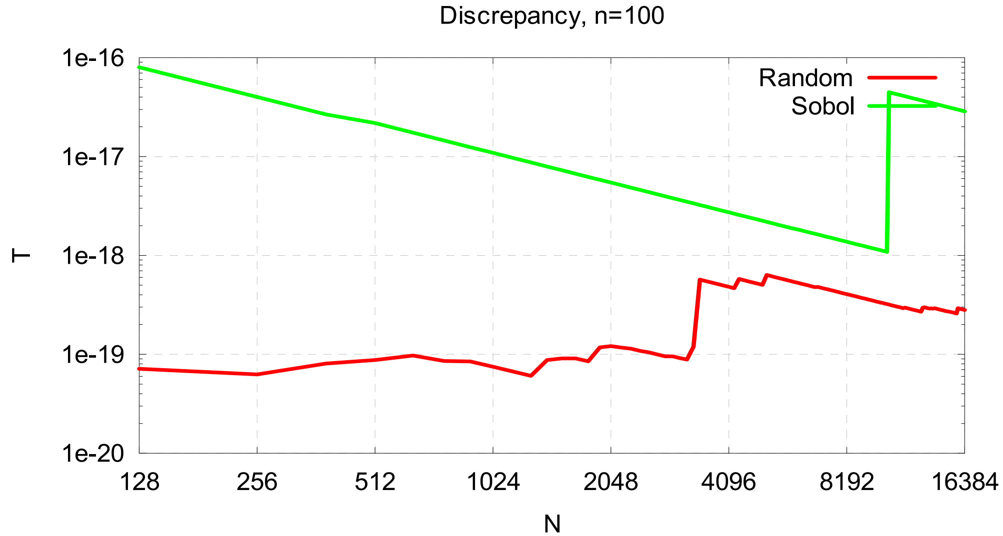




Imperial College London

Discrepancy
II. High Dimensions





MC in high-dimensions has smaller discrepancy

Imperial College London

Are QMC efficient for high dimensional problems

$$\varepsilon_{QMC} = \frac{O(\ln\ N)^n}{N}$$

Assymptotically $\varepsilon_{QMC} \sim O(1/N)$

but ε_{QMC} increseas with N until $N \approx \exp(n)$

$n = 50, N \approx 5\ 10^{21}$ – not achievable for practical applications

"For high-dimensional problems (n > 12),

QMC offers no practical advantage over Monte Carlo"

(Bratley, Fox, and Niederreiter (1992)) ?!

Is MC more efficient for high-dimensional problems than QMC ?

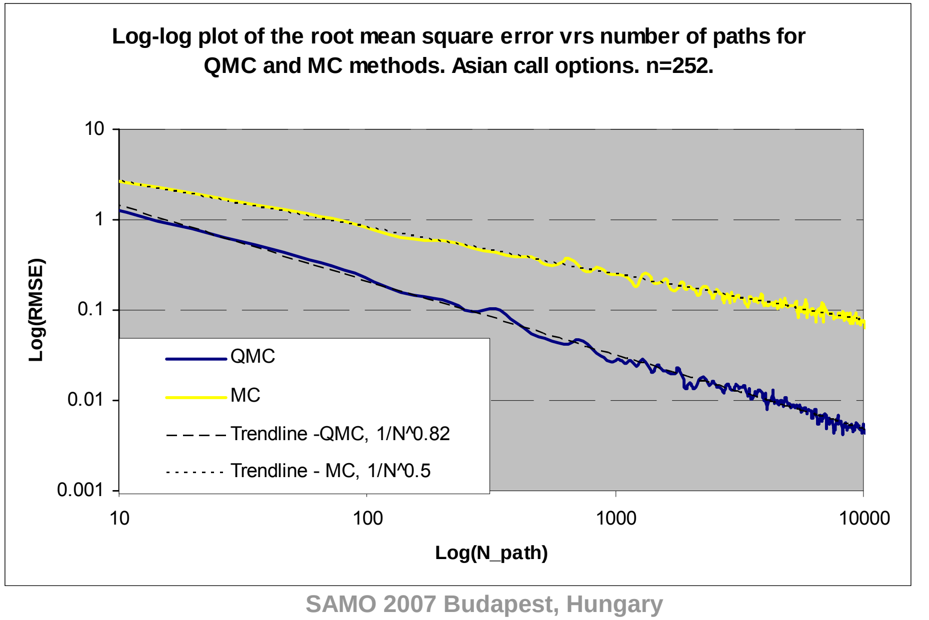
- **Pros:**

  MC in high-dimensions has smaller discrepancy

  Some studies show degradation of the convergence rate of QMC methods in high-dimensions to O(1/√N)

- **Cons:**

  Huge success of QMC methods in finance: QMC methods were proven to be much more efficient than MC even for problems with thousands of variables



Imperial College London

Sensitivity Indices (SI)

Consider a model

x is a vector of input variables

Y is the model output.

$$Y = f(x)$$
$$x = (x_1, x_2, \ldots, x_k)$$
$$0 \le x_i \le 1$$

ANOVA decomposition (HDMR):

$$Y = f(x) = f_0 + \sum_{i=1}^{k} f_i(x_i) + \sum_i \sum_{j>i} f_{ij}(x_i, x_j) + \ldots + f_{1,2,\ldots,k}(x_1, x_2, \ldots, x_k),$$
$$\int_0^1 f_{i_1 \cdots i_s}(x_{i_1}, ,\ldots, x_{i_s}) \; dx_{i_k} = 0, \;\; \forall k, \; 1 \le k \le s$$

Variance decomposition:

$$\sigma^2 = \sum_i \sigma_i^2 + \sum_{i,j} \sigma_{ij}^2 + \ldots \sigma_{1,2,\ldots,n}^2$$

Sobol' SI:

$$1 = \sum_{i=1}^{k} S_i + \sum_{i<j} S_{ij} + \sum_{i<j<l} S_{ijl} + \ldots + S_{1,2,\ldots,k}$$

Effective dimension

Let $|u|$ be a cardinality of a set of variables u.

The effective dimension of $f(x)$ in the superposition sense is the smallest integer d_S such that

$$\sum_{0<|u|<d_S} S_u \geq (1-\varepsilon), \ \ \varepsilon << 1$$

It means that $f(x)$ is almost a sum of d_S-dimensional functions.

The function $f(x)$ has effective dimension in the truncation sense d_T if

$$\sum_{u \subseteq \{1,2,...,d_T\}} S_u \geq (1-\varepsilon), \ \ \varepsilon << 1$$

Example: $f(x) = \sum_{i=1}^{n} x_i \rightarrow d_S = 1, \ d_T = n$

d_S does not depend on the order in which the input variables are sampled,

d_T - depends on the order $\rightarrow$ by reodering variables d_T can be reduced

$x=(x_1,\ldots,x_n)$

Approximation errors

For many problems only low order terms in the ANOVA decomposition are important $d \ll n$

$$h(x) = f_0 + \sum_{s=1}^{d} \sum_{i_1<\ldots<i_s}^{s} f_{i_1\ldots i_s}(x_{i_1},\ldots,x_{i_s})$$

Consider an approximation error $\delta(f,h)$

$$\delta(f,h) = \frac{1}{\sigma}\int [f(x) - h(x)]^2 dx$$

Theorem 1: $\delta(f,h) = 1 - \sum_{s=1}^{d} \sum_{i_1<\ldots<i_s}^{s} S_{i_1\ldots i_s}(x_{i_1},\ldots,x_{i_s})$

Link between an approximation error and

effective dimension in superposition sense $d_S = d:\ \delta(f,h) \le \varepsilon$

$x=(y,z):\ y=(x_1,\ldots,x_d), z=(z_{d+1},\ldots,x_n)$

Set of variables z can be regarded as not important

if $S_z^{tot} \ll 1$ If $z=z_0$ and $f(x) \approx f(y,z_0)$

Consider an approximation error $\delta(z_0)$

$$\delta(z_0) = \frac{1}{D}\int [f(x) - f(y,z_0)]^2 dx$$

Theorem 2: $\mathbf{E}\delta(z_0) = 2S_z^{tot}$

Link between an approximation error and effective dimension in truncation sense $d_T = d:$

$$\mathbf{E}\delta(z_0) \le 2\varepsilon$$

Classification of functions

Type A. Variables are not equally important

$$\frac{S_y^T}{n_y} >> \frac{S_z^T}{n_z} \leftrightarrow d_T << n$$

Type B,C. Variables are equally important

$$S_i^T \approx S_j^T \leftrightarrow d_T \approx n$$

Type B.
Dominant low order indices

$$\sum_{i=1}^{n} S_i \approx 1 \leftrightarrow d_S << n$$

Type C. Dominant higher order indices

$$\sum_{i=1}^{n} S_i << 1 \leftrightarrow d_S \approx n$$

Imperial College London

Sensitivity indices for type A functions

$$\frac{S_y^T}{n_y} >> \frac{S_z^T}{n_z} \leftrightarrow d_T << n$$

Index	Function $f(\underline{x})$	Ref.	Measure	Analytical values	Numerical Values $n=2$	$n=10$	$n=100$
1A	$\sum_{i=1}^{n}(-1)^i \prod_{j=1}^{i} x_j$	[Bratley,1992]	$\frac{S_1}{S_1^{tot}}$	$\frac{12}{27}\frac{\left(1-\left(-\frac{1}{2}\right)^n\right)^2}{\frac{1}{2}-\frac{4}{5}\left(-\frac{1}{2}\right)^n+\frac{3}{10}\left(\frac{1}{3}\right)^n}$	0.75	0.89	0.89
			$\sum S_i$	[-]	0.86	0.89	0.89
2A	$\prod_{i=1}^{n}\frac{\lvert 4x_i-2\rvert+a_i}{1+a_i}$,	[Saltelli,1995]	$\frac{S_1}{S_1^{tot}}$	$\frac{(1+D)^{(2-n)}}{(1+C)}$	-	-	0.0004
	$a_1 = a_2 = 0$,		$\sum S_i$	$\frac{2C+(n-2)D}{(1+C)^2(1+D)^{(n-2)}-1}$	-	-	0.003
	$a_3 = \ldots = a_{100} = 6.52$						
				$C = \frac{1}{3(a_1+1)^2}$, $D = \frac{1}{3(a_3+1)^2}$			

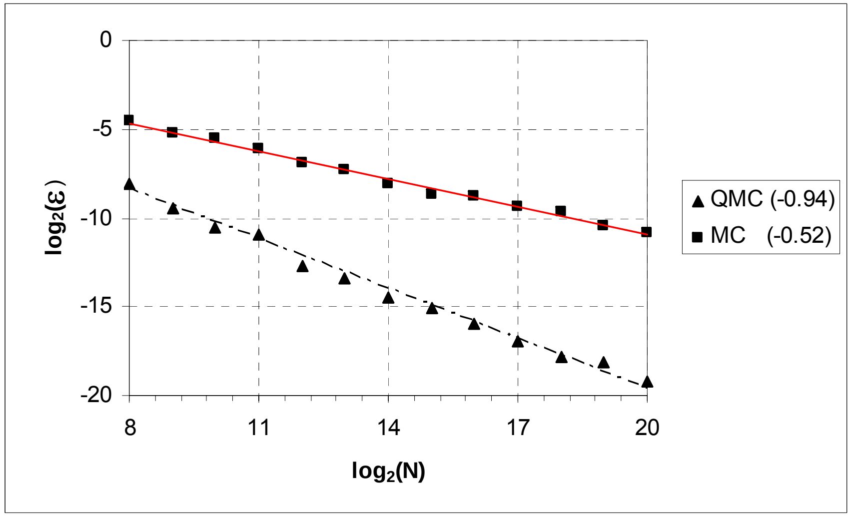
Integration error vs. N. Type A

(a) $f(x) = \sum_{j=1}^{n}(-1)^i \prod_{j=1}^{i} x_j$, $n = 360$, (b) $f(x) = \prod_{i=1}^{s} |4x_i - 2| / (1 + a_i)$, $n = 100$

$$\varepsilon = \left(\frac{1}{K}\sum_{k=1}^{K}(I - I_N^k)^2\right)^{1/2}$$

$$\varepsilon \sim N^{-\alpha}, \; 0 < \alpha < 1$$

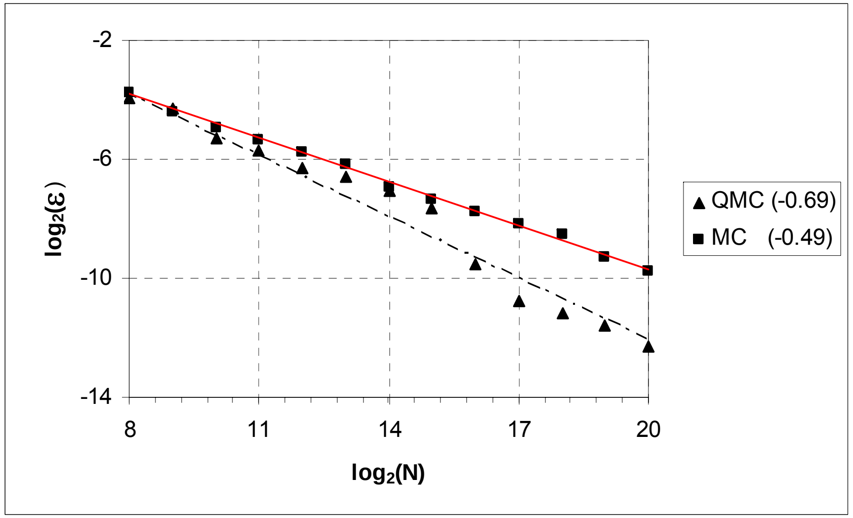
(a)



$$\frac{S_y^T}{n_y} >> \frac{S_z^T}{n_z} \leftrightarrow d_T << n$$

$$S_{\{1,2\}}^T = 0.94, S_{\{3,4,\ldots 360\}}^T = 0.1$$

(b)



$$S_{\{1,2\}}^T = S_{\{3,4,\ldots 100\}}^T = 0.64$$

Sensitivity indices for type B functions

Dominant low order indices

$$\sum_{i=1}^{n} S_i \approx 1 \leftrightarrow d_S << n$$

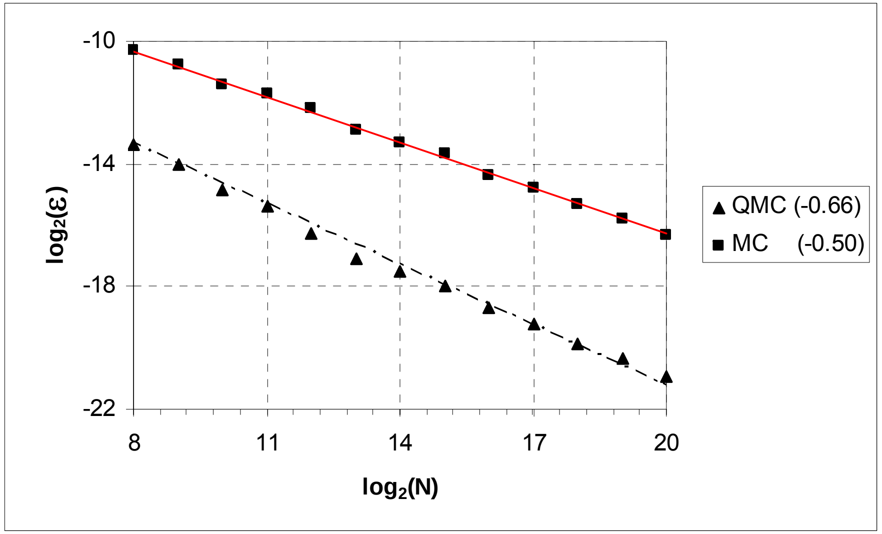
Index	Function $f(\underline{x})$	Ref.	Measure	Analytical values	Numerical Values $n=2$	$n=10$	$n=100$
1B	$\prod_{i=1}^{n} \frac{n-x_i}{n-0.5}$	[Levitan,1988]	$\frac{S_1}{S_1^{tot}}$	$\left(\frac{1}{1+\frac{1}{12(n-0.5)^2}}\right)^{n-1}$	0.96	0.992	0.999
			$\sum S_i$	$\frac{n}{12(n-\frac{1}{2})^2\left[\left(1+\frac{1}{12(n-\frac{1}{2})^2}\right)^n-1\right]}$	0.981	0.995	0.999
2B	$\left(1+\frac{1}{n}\right)^n \prod_{i=1}^{n} \sqrt[n]{x_i}$	[Levitan,1988]	$\frac{S_1}{S_1^{tot}}$	$\left(1+\frac{1}{n^2+2n}\right)^{1-n}$	0.88	0.93	0.99
			$\sum S_i$	$\frac{n}{(n^2+2n)\left[\left(1+\frac{1}{n^2+2n}\right)^n-1\right]}$	0.941	0.963	0.995
3B	$\prod_{i=1}^{n} \frac{\lvert 4x_i-2\rvert+a_i}{1+a_i}$,	[Saltelli,1995]	$\frac{S_1}{S_1^{tot}}$	$\left(1+\frac{1}{3(a_i+1)^2}\right)^{(1-n)}$	0.99	0.95	0.55
$a_i = 6.52$			$\sum S_i$	$\frac{n \cdot \frac{1}{3(a_i+1)^2}}{\left(1+\frac{1}{3(a_i+1)^2}\right)^n-1}$	0.99	0.97	0.74

Integration error vs. N. Type B

Dominant low order indices

$$\sum_{i=1}^{n} S_i \approx 1 \leftrightarrow d_S << n$$

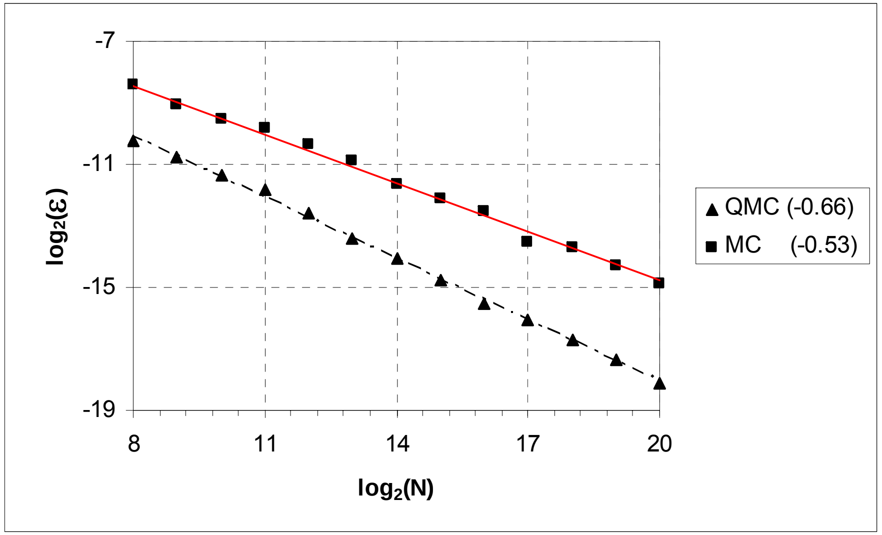
(a)



$$f(x) = \prod_{i=1}^{n} \frac{n - x_i}{n - 0.5}$$

$$n = 360$$

(b)



$$f(x) = \prod_{i=1}^{n} (1 + 1/n) x_i^{1/n}$$

$$n = 360$$

Imperial College London

Sensitivity indices for type C functions

Dominant higher order indices

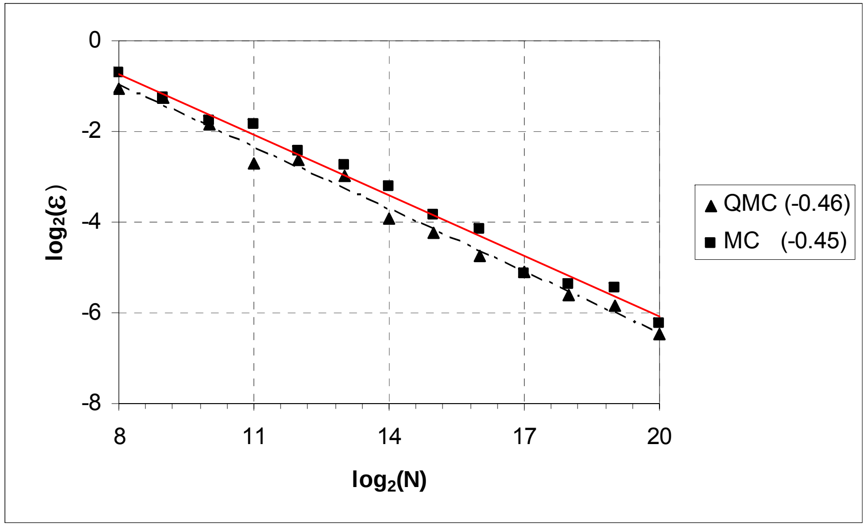
$$\sum_{i=1}^{n} S_i << 1 \leftrightarrow d_S \approx n$$

Index	Function $f(\underline{x})$	Ref.	Measure	Analytical values	Numerical Values $n=2$	$n=10$	$n=100$
1C	$\prod_{i=1}^{n} \lvert 4x_i-2 \rvert$	[Bratley,1988]	$\frac{S_1}{S_1^{tot}}$	$\left(\frac{4}{3}\right)^{1-n}$	0.75	0.075	$4.3\cdot 10^{-13}$
			$\sum S_i$	$\frac{n}{3\left(\left(\frac{4}{3}\right)^n-1\right)}$	0.86	0.20	$1.06\cdot 10^{-11}$
2C	$(2)^n \prod_{i=1}^{n} x_i$	[-]	$\frac{S_1}{S_1^{tot}}$	$\left(\frac{3}{4}\right)^{(n-1)}$	0.75	0.075	$4.28\cdot 10^{-13}$
			$\sum S_i$	$\frac{n}{3\left(\left(\frac{4}{3}\right)^n-1\right)}$	0.86	0.20	$1.07\cdot 10^{-11}$
3C	$\prod_{i=1}^{n} \frac{\lvert 4x_i-2 \rvert+a_i}{1+a_i}$,	[Saltelli,1995]	$\frac{S_1}{S_1^{tot}}$	$\left(1+\frac{1}{3(a_i+1)^2}\right)^{(1-n)}$	0.92	0.48	0.00036
	$a_i=1$		$\sum S_i$	$\frac{n\cdot\frac{1}{3(a_i+1)^2}}{\left(1+\frac{1}{3(a_i+1)^2}\right)^n-1}$	0.96	0.67	0.003

The integration error vs. N. Type C

Dominant higher order indices: $\sum_{i=1}^{n} S_i << 1 \leftrightarrow d_S \approx n$

(a)

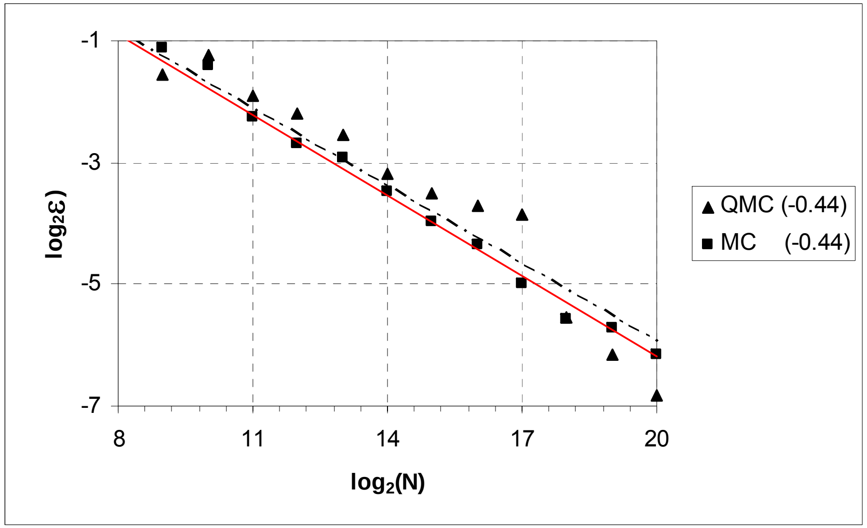


$$f(x) = \prod_{i=1}^{n} \frac{|4x_i - 2| + a_i}{1 + a_i}, a_i = 0$$

$$\rightarrow \prod_{i=1}^{n} |4x_i - 2|$$

$$n = 20$$

(b)



$$f(x) = (1/2)^{1/n} \prod_{i=1}^{n} x_i$$

$$n = 20$$

Imperial College London

Application of GSA to path dependent integrals

$$I = \int_C F[x(t)] d_W x,$$

$x(t)$ – continuous in $0 \le t \le T$, $x(0) = x_0$

$I = E(F[\xi(t)])$, $\xi(t)$ – random Wiener processes (a Brownian motion)

Monte Carlo approach: to construct many random paths $\xi(t)$, evaluate and average results

Practical applications:

Share price follows geometrical Brownian motion:

$$dS = \mu S dt + \sigma S d\xi, \quad d\xi = z(dt)^{1/2}, z \sim N(0,1)$$

$$S(t) = S_0 \exp[(\mu - \frac{1}{2}\sigma^2)t + \sigma\xi(t))], \quad \xi(t) \text{ – Wiener path}$$

Imperial College London

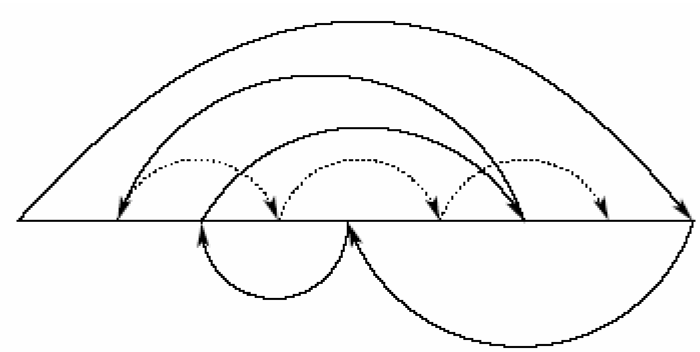
Approximations of path dependent integrals using standard and Brownian bridge discretizations

SDE: $$d\xi = z\sqrt{dt}, \quad z \sim N(0,1)$$

Standard algorithm:

$$\xi(t_{i+1}) = \xi(t_i) + \sqrt{\Delta t}\, z_{i+1}, \quad \Delta t = T/n, \quad 0 \le i \le n-1$$

Brownian bridge algorithm:



$$\xi(T) = \xi_0 + \sqrt{T}\, z_1,$$

$$\xi(T/2) = \frac{1}{2}(\xi(T) + \xi_0) + \frac{1}{2}\sqrt{T}\, z_2,$$

$$\xi(T/4) = \frac{1}{2}(\xi(T/2) + \xi_0) + \frac{1}{2}\sqrt{T/2}\, z_3,$$

$$\xi(3T/4) = \frac{1}{2}(\xi(T/2) + \xi(T)) + \frac{1}{2}\sqrt{T/2}\, z_4,$$

$$\vdots$$

$$\xi((n-1)T/n) = \frac{1}{2}(\xi((n-2)T/n) + \xi(T)) + \frac{1}{2}\sqrt{2T/n}\, z_n.$$

Imperial College London

Global sensitivity analysis of F_n

Functional: $$F[\xi(t)] = \int_0^T \xi^2(t)dt.$$

Numerical approximation: $$F_n = \sum_i a_i z_i^2 + \sum_{i<j} a_{ij} z_i z_j = F_n(z_1, ..., z_n)$$

a_i, a_{ij} depend on *n* and on the type of the approximation

Expectation: $$I_n = E(F_n) = \sum_i a_i$$

ANOVA decomposition: $$F_n = I_n + \sum_i a_i (z_i^2 - 1) + \sum_{i<j} a_{ij} z_i z_j.$$

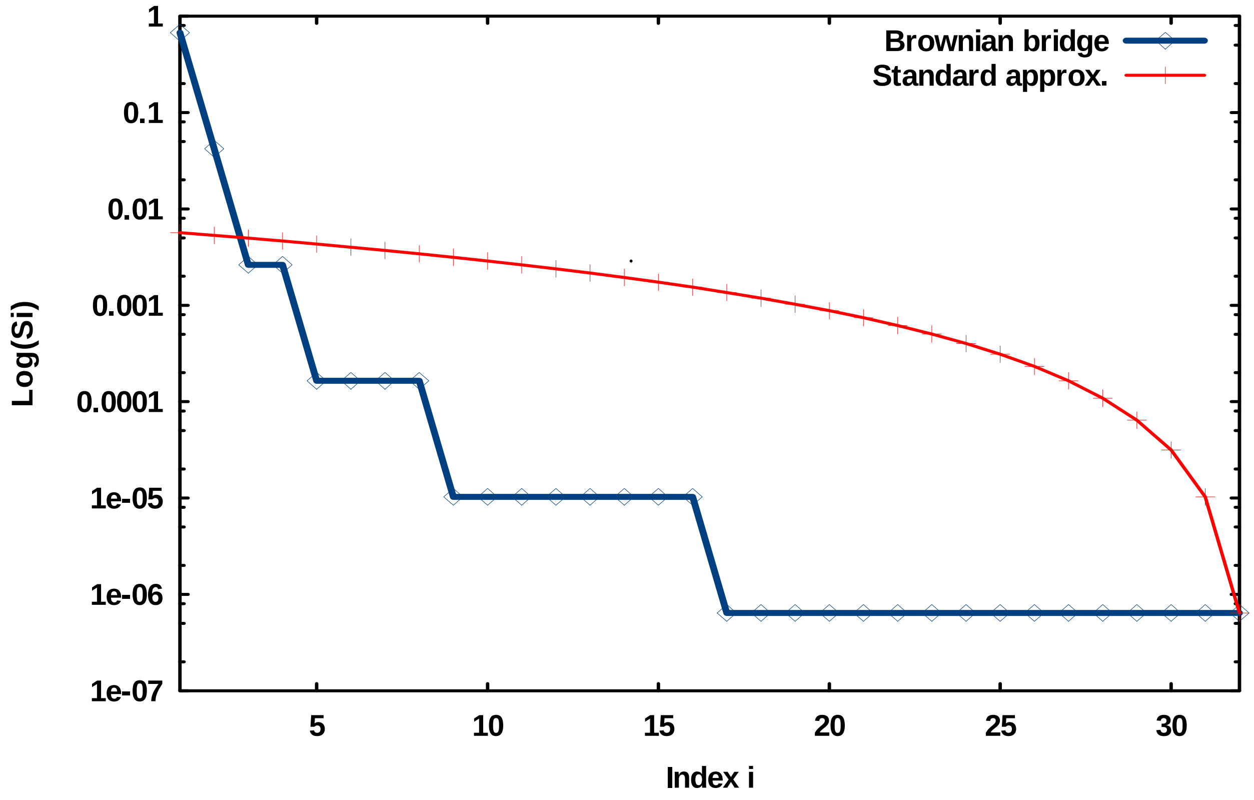
Variance: $$D(F_n) = 2\sum_i a_i^2 + \sum_{i<j} a_{ij}^2.$$

Sensitivity indices: $$S_i = 2a_i^2 / D(F_n),$$

$$S_{ij} = a_{ij}^2 / D(F_n)$$

Imperial College London

First order sensitivity indices S_i versus index number i, n=32.



GSA of two algorithms at different *n*

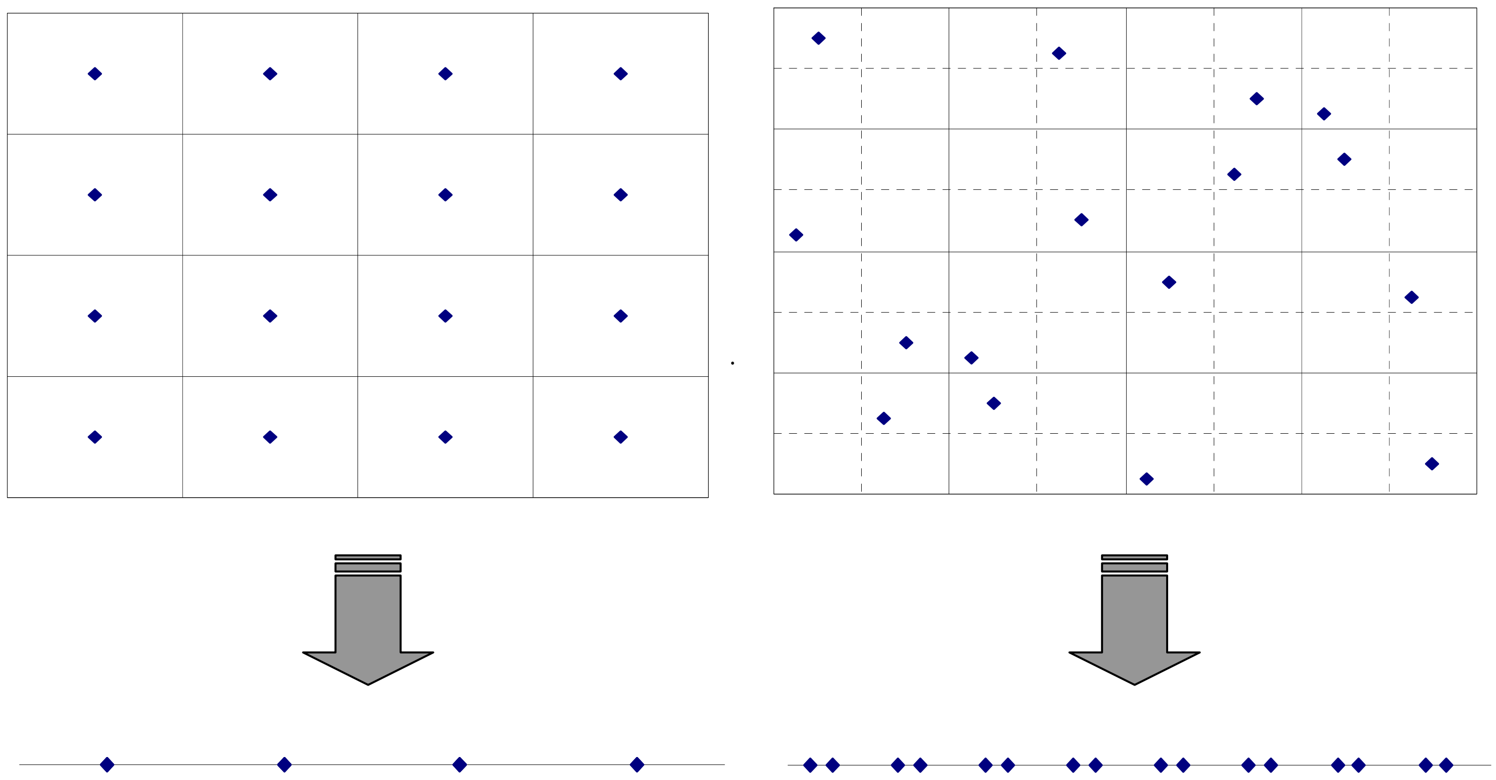
Function	Measure	Numerical Values n=8	n=32
Brownian Bridge approximation $\int_0^1 x^2(t)dt$	$\frac{S_1}{S_1^{tot}}$	0.72	0.70
	$\sum S_i$	0.72	0.72
	$\sum S_{ij}$	0.28	0.28
	$S_z^{tot}(d_T)$	0.09	0.10
	d_T	2	2
	d_S	2	2
Standard approximation $\int_0^1 x^2(t)dt$	$\frac{S_1}{S_1^{tot}}$	0.17	0.06
	$\sum S_i$	0.25	0.062
	$\sum S_{ij}$	0.77	0.94
	$S_z^{tot}(d_T)$	0.05	0.09
	d_T	6	22
	d_S	2	2

Standard approximation: the effective dimension $d_T \approx \frac{3}{4} n$

Brownian Bridge approximation: both effective dimensions are close to 2

What is the optimal way to arrange N points in two dimensions?

Regular Grid   Sobol’ Sequence

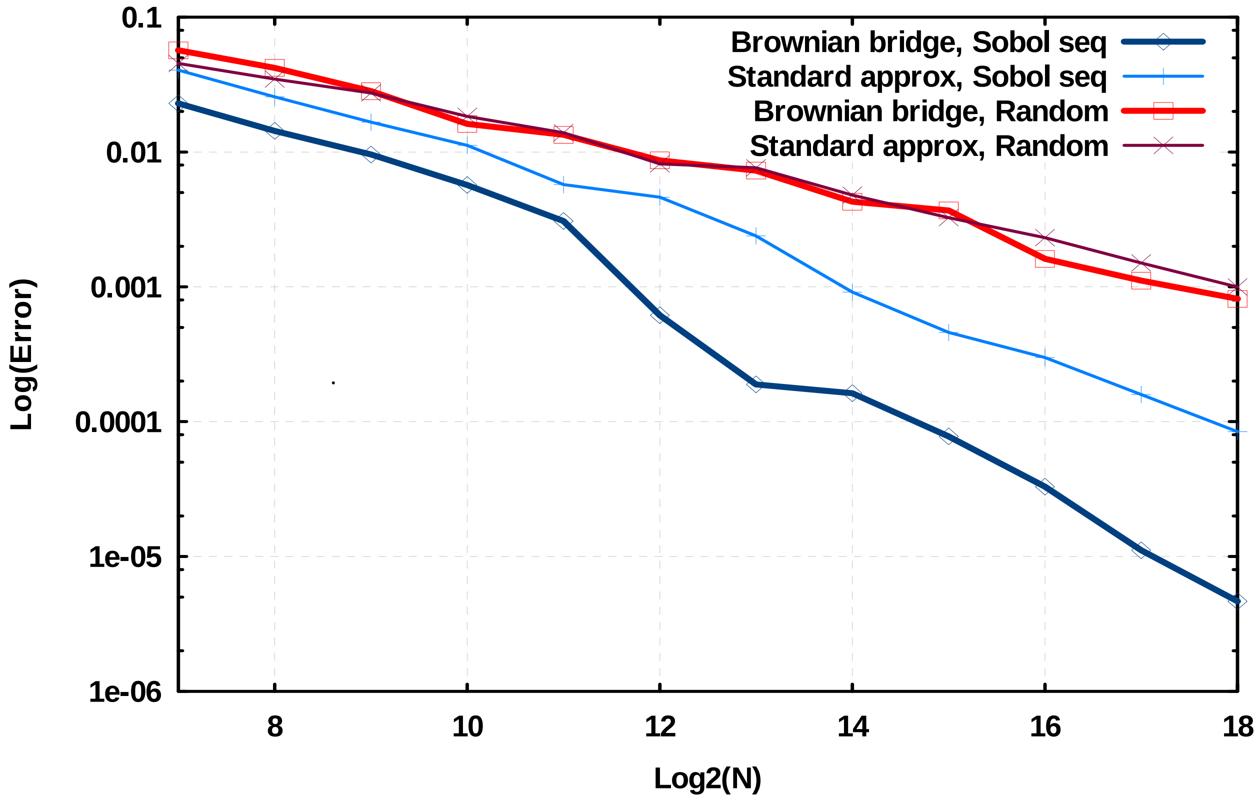


Low dimensional projections of low discrepancy sequences are better distributed than higher dimensional projections

Convergence curves at n=64

$$\varepsilon = \left(\frac{1}{K} \sum_{k=1}^{K} (I_n - I_{n,N}^{k})^2 \right)^{1/2}$$

$$MC \sim O(1/\sqrt{N})$$
$$QMC \sim O(1/N)$$



Brownian Bridge discretizations - the effective dimension reduction technique

Original vrs Improved formulae for evaluation of Sobol' Sensitivity Indices

$$S_y = \frac{\int_0^1 f(y,z)\, f(y,z')\; dydzdz' - f_0^2}{\int_0^1 f^2(y,z)\, dydz - f_0^2}$$

for small indices $S_y << 1$

$$\int_0^1 f(y,z)\, f(y,z')\; dydzdz' \approx f_0^2$$

$\rightarrow$ loss of accuracy

Notice that $f_0^2 = \int_0^1 f(y,z) dydz \int_0^1 f(y',z')\; dy'dz'$

$$S_y = \frac{\int_0^1 f(y,z)\left[f(y,z') - f(y',z') \right] dydzdz'}{\int_0^1 \left[f(y,z) - f_0 \right]\left[f(y,z) + f_0 \right] dydz}$$

$\rightarrow$ much more accurate

Requires $N(n+2)$ model evalution $\leftrightarrow$ original Sobol' formulas $N(2n+1)$

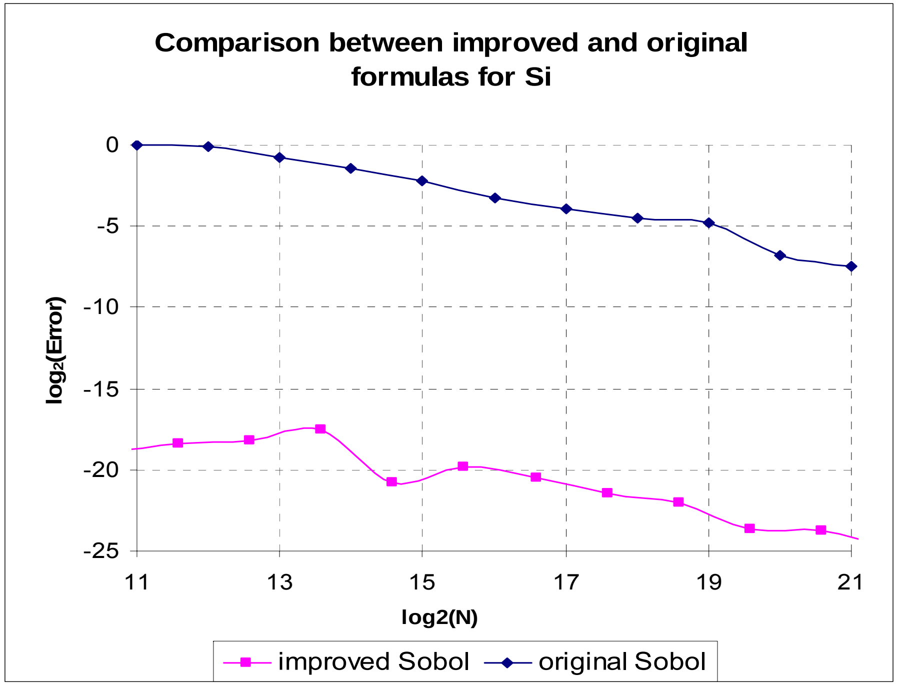
The same model evaluations can be used for computing second order indices

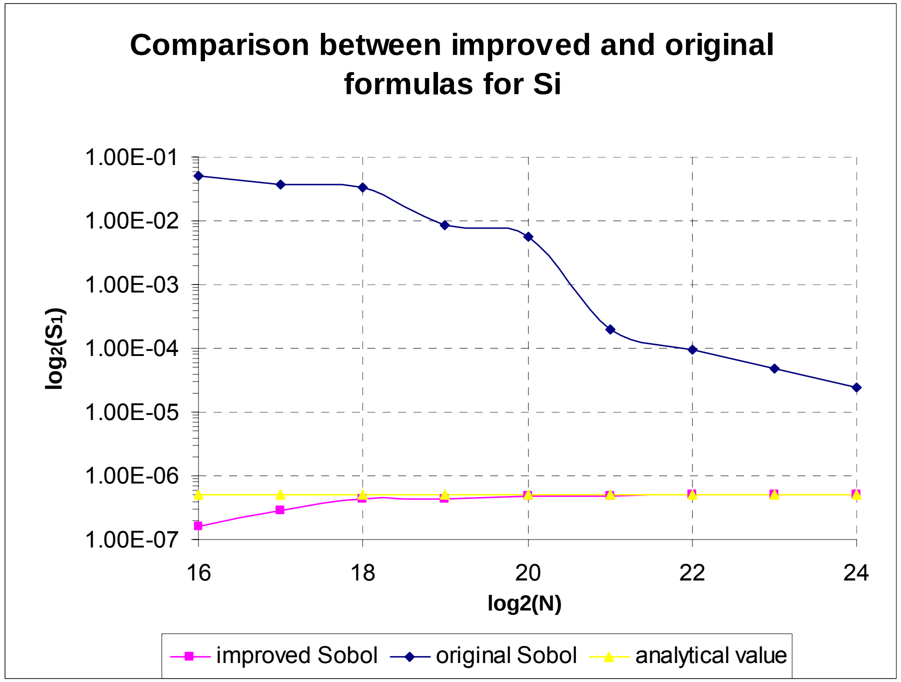
Imperial College London

Improved formula for Sobol' Sensitivity Indices

$$f(x)=\sum_{i=1}^{n} ix_i\,,\quad S_i=S^T=\frac{6}{n(n+1)(2n+1)}$$

$$n=180,\quad S_1=5.1\,10^{-7}$$





Imperial College London

Optimal experimental design (OED) for parameter estimation

Find values of experimentally manipulable variables (controls) and the time sampling strategy for a set of N_{exp} experiments which provides maximum information for the subsequent parameter estimation problem

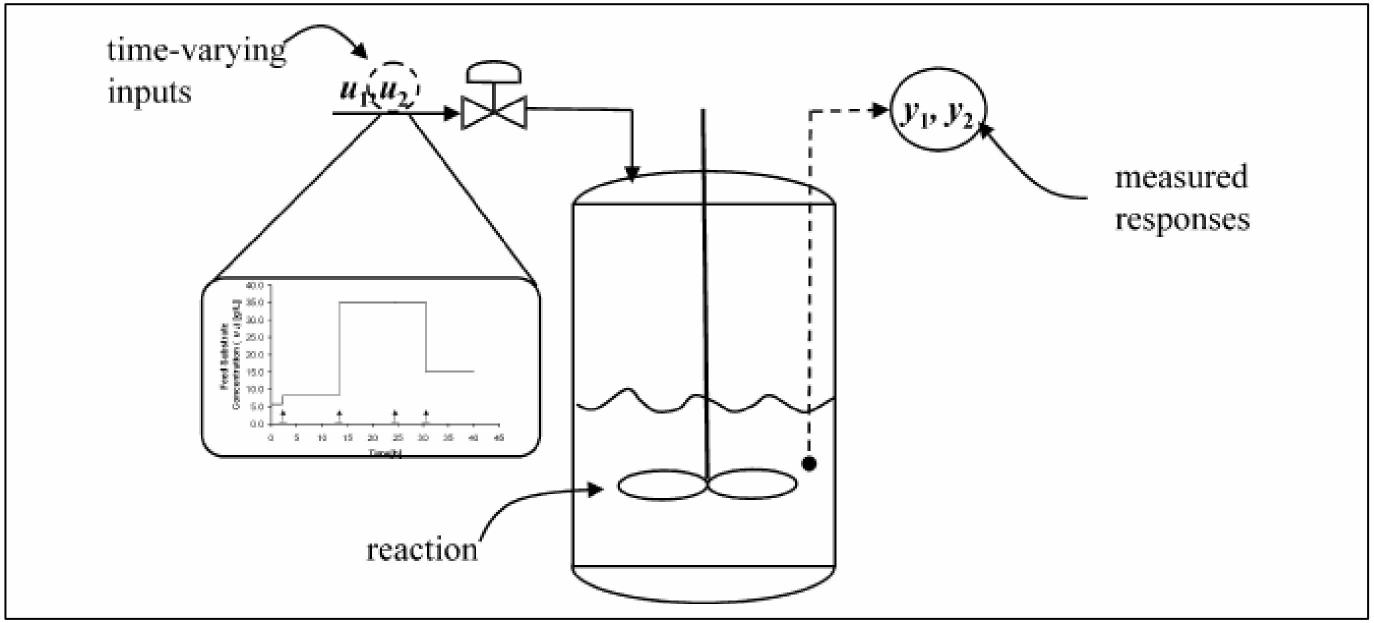
- *subject to:*
  - *System dynamics (ODEs, DAEs)*
  - *Other algebraic constraints*
  - *Upper and lower bounds:*

$$u^L \leq u \leq u^U$$

Non-linear programming problem (NLP) with partial differential-algebraic (PDAEs) constraints

Imperial College London

Case study: fed-batch reactor



- **Biomass:** $\frac{dy_1}{dt} = (r_m - u_1)y_1 - p_1 y_1$
- **Substrate:** $\frac{dy_2}{dt} = \frac{r_m y_1}{p_2} + u_1(u_2 - y_2)$
- **Reaction rate:** $r_m = \frac{0.5 y_2}{0.5 + y_2}$

- **Parameters to be estimated: p_1, p_2**

  $0.05 < p_1 < 0.98$, $0.05 < p_2 < 0.98$

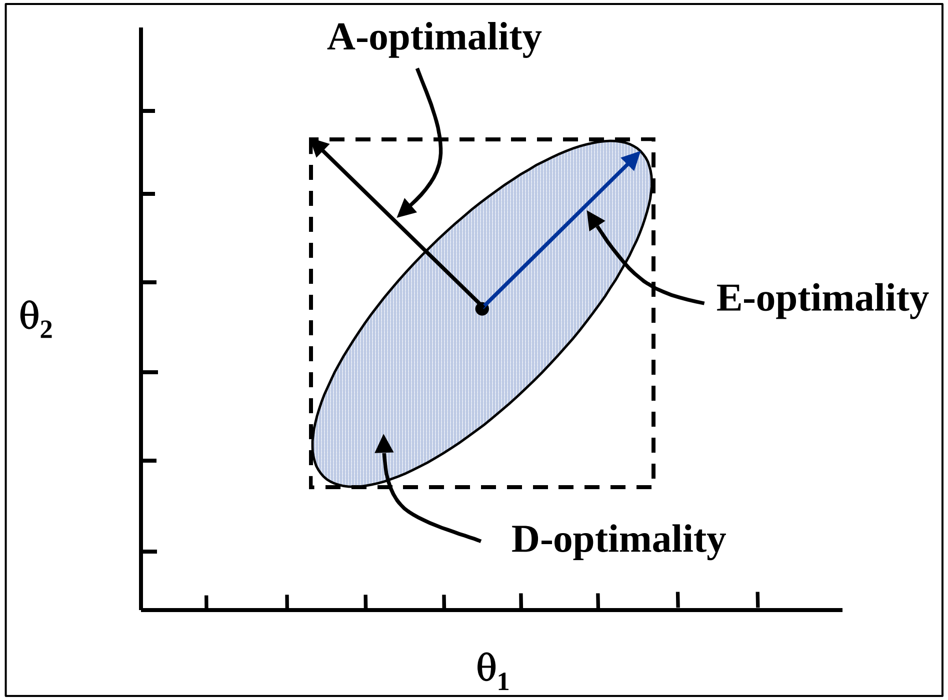
- **Control variables: u_1, u_2**
  - ***Dilution factor: $0.05 < u_1 < 0.5$***
  - ***Feed substrate concentration:***

    $5 < u_2 < 50$

OED traditional approach

- ***Fisher Information Matrix (FIM) based criteria:***

$$FIM = \sum_{i=1}^{N} \left(\frac{\partial y}{\partial p}(t_i) \right)^T W_i \left(\frac{\partial y}{\partial p}(t_i) \right)$$



A criterion = $\min\left[trace\left(FIM^{-1}\right)\right]$

D criterion = $\max\left[\det(FIM)\right]$

E criterion = $\max\left[\lambda_{\min}(FIM)\right]$

Modified-E criterion = $\min\left[\frac{\lambda_{\max}(FIM)}{\lambda_{\min}(FIM)}\right]$

Main drawback: based on local SI $\Rightarrow$ non-realistic linear and local assumptions

Imperial College London

Parametric GSA

Nonlinear dynamic model: $\vec{Y} = f(\vec{p}, \vec{u}, t)$

$\vec{p}$ - uncertain parameters,

$\vec{u}$ - control variables,

t - time

Find $S_i(\vec{u}, t), S_i^T(\vec{u}, t)$ – depend on parameters !

Solve: $\max_{\vec{u}} F(S_i(\vec{u}, t)) \rightarrow$

OED $\vec{u}*$ for parameter estimation

- Optimal experimental design: identification of a set of experiments with conditions that deliver measurement data that are the most sensitive to the unknown parameters

Application of Parametric GSA for parameter optimization

$$FIM = \sum_{i=1}^{N} \left[Q\left(\vec{u}, t_i\right)^T W_i Q_i\left(\vec{u}, t_i\right) \right]$$

$$Q(t_i) = \begin{bmatrix} \frac{\partial y_1}{\partial p_1}\left(\vec{u}, t_i\right) & \frac{\partial y_1}{\partial p_2}\left(\vec{u}, t_i\right) & \cdots & \frac{\partial y_1}{\partial p_p}\left(\vec{u}, t_i\right) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_s}{\partial p_1}\left(\vec{u}, t_i\right) & \frac{\partial y_s}{\partial p_2}\left(\vec{u}, t_i\right) & \cdots & \frac{\partial y_s}{\partial p_p}\left(\vec{u}, t_i\right) \end{bmatrix}$$

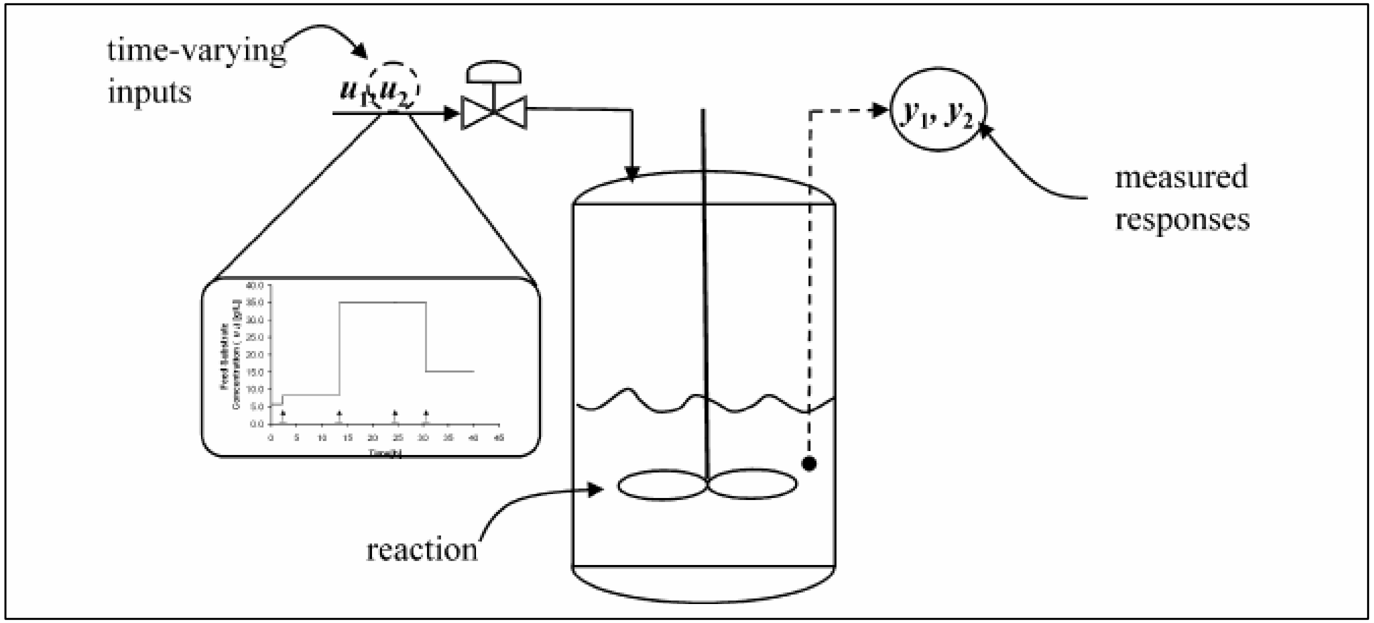
$$GSIM = \sum_{i=1}^{N} \left[Q\left(\vec{u}, t_i\right)^T W_i Q_i\left(\vec{u}, t_i\right) \right]$$

$$Q(t_i) = \begin{bmatrix} S_{1,1}\left(\vec{u}, t_i\right) & S_{1,2}\left(\vec{u}, t_i\right) & \cdots & S_{1,p}\left(\vec{u}, t_i\right) \\ S_{2,1}\left(\vec{u}, t_i\right) & S_{2,2}\left(\vec{u}, t_i\right) & \cdots & S_{2,p}\left(\vec{u}, t_i\right) \\ \vdots & \vdots & \ddots & \vdots \\ S_{s,1}\left(\vec{u}, t_i\right) & S_{s,2}\left(\vec{u}, t_i\right) & \cdots & S_{s,p}\left(\vec{u}, t_i\right) \end{bmatrix}$$

Main advantage: based on global SI $\Longrightarrow$ allows to consider a range of values for the parameters to be estimated

- ***objective function:*** $\max_{\vec{u}} \det(GSIM)$
- ***Application of Global Optimization method***

Case study: fed-batch reactor



- **Biomass:** $\frac{dy_1}{dt} = (r_m - u_1)y_1 - p_1 y_1$
- **Substrate:** $\frac{dy_2}{dt} = \frac{r_m y_1}{p_2} + u_1(u_2 - y_2)$
- **Reaction rate:** $r_m = \frac{0.5 y_2}{0.5 + y_2}$

- ***Parameters to be estimated:*** p_1, p_2

  $0.05 < p_1 < 0.98$, $0.05 < p_2 < 0.98$

- ***Control variables:*** u_1, u_2
  - ***Dilution factor:*** $0.05 < u_1 < 0.5$
  - ***Feed substrate concentration:***

    $5 < u_2 < 50$

Optimal Experimental Design

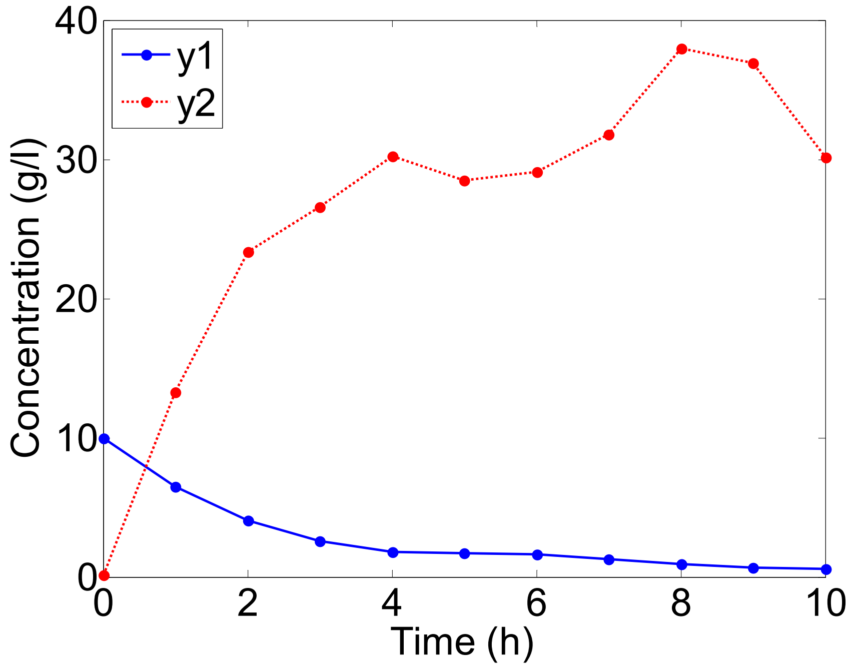
- ***Problem constraints:***

  Experiment duration: 10 h

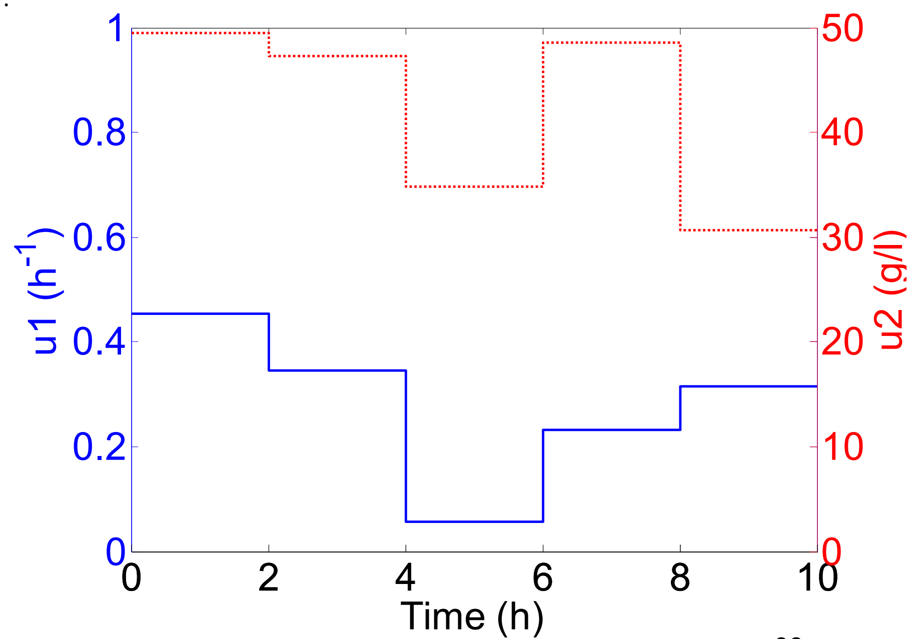
  Number of measurement times: 10

  Controls varied every 2 hours

- ***Results:***



Optimal input profile for $\mathbf{u_1}$ and $\mathbf{u_2}$:



Imperial College London

Setting of the Parameter Estimation Problem

- **Steps to find p:**

  - *Take experimental or generated pseudo-experimental points* $\tilde{y}$
  - *Maximum likelihood optimization*

$$J_{ml}(p) = \prod_{i=1}^{NE} \prod_{j=1}^{NV_i} \prod_{k=1}^{NM} \left(2\pi\sigma_{ijk}^2\right)^{-1/2} \exp\left\{-\frac{1}{2}\left[\frac{y_{ijk}(p) - \tilde{y}_{ijk}}{\sigma_{ijk}}\right]^2\right\}$$

p: set of parameters to be estimated	$y_{ki}(p)$: model prediction
σ_{ki}^2 : measurements variance	$\tilde{y}_{ki}$: experimental measures

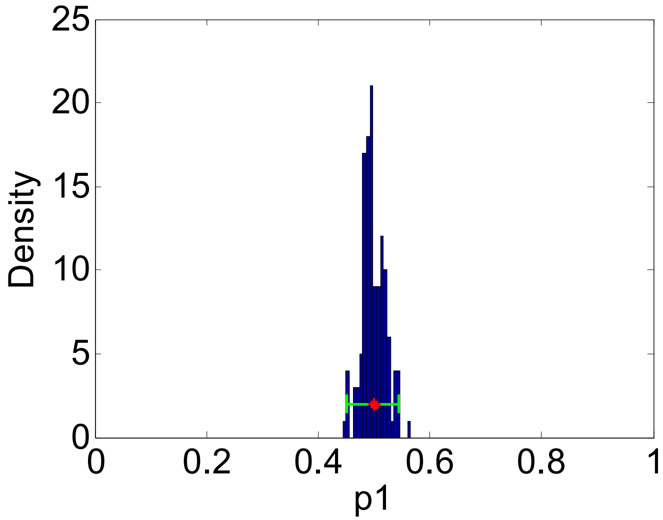
- **subject to:**
  - *System dynamics (ODEs, DAEs)*
  - *Other algebraic constraints*
  - *Upper and lower bounds:*

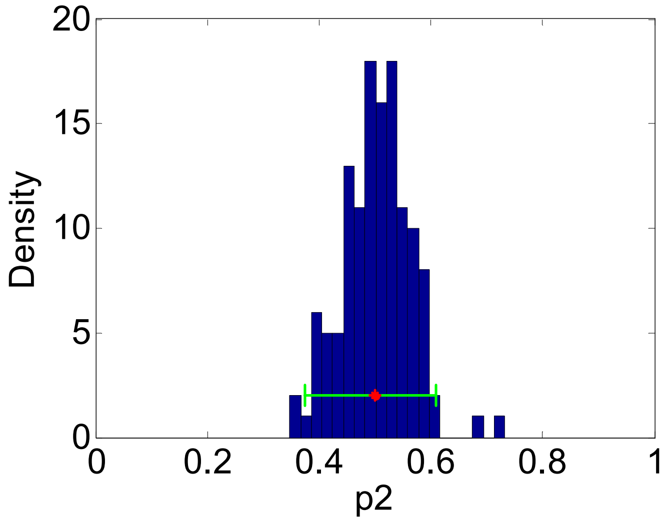
$$p^L \leq p \leq p^U$$

Non-linear programming problem (NLP) with partial differential-algebraic (PDAEs) constraints

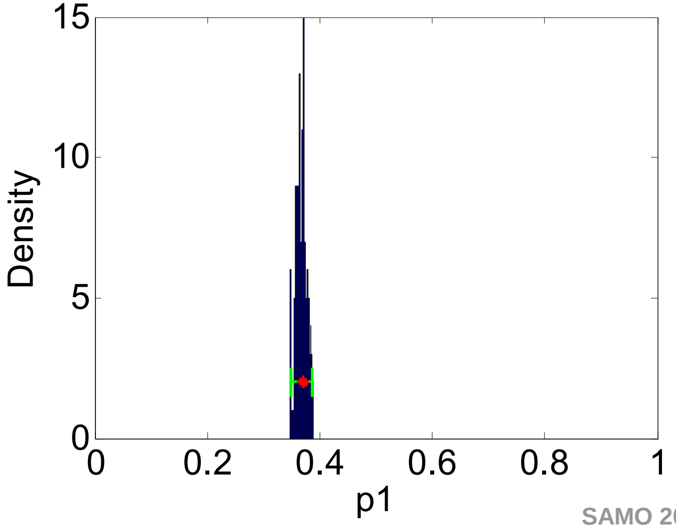
Results of parameter estimation

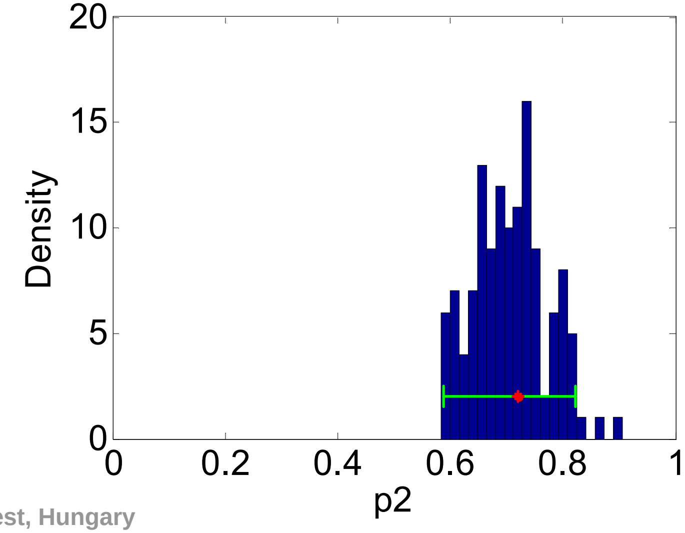
- $p_1 = 0.5 \pm 0.05$, $p_2 = 0.5 \pm 0.11$





- $p_1 = 0.37 \pm 0.02$, $p_2 = 0.72 \pm 0.12$





Imperial College London

Summary

Global Sensitivity Analysis can be successfully used for measuring the effectiveness of Quasi-Monte Carlo methods

Quasi Monte Carlo methods based on Sobol' sequences outperform Monte Carlo even in high dimensions

By reducing the effective dimension the efficiency of Quasi Monte Carlo methods can be further improved

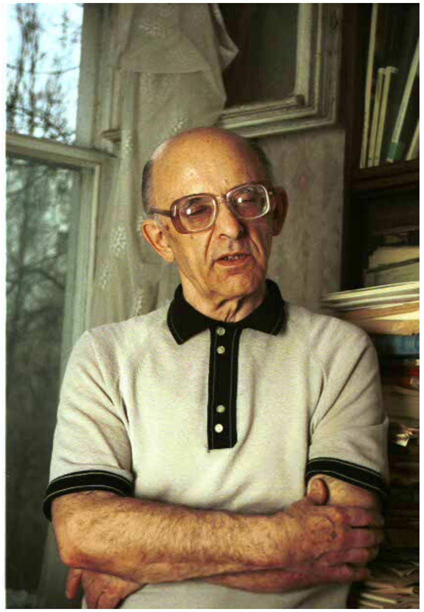
Application of global SI to OED results in the reduction of the required experimental work and the increased accuracy of parameter estimation

Imperial College London

Thank to SAMO organizing committee for inviting me to the conference !

Acknowledgments

Prof. Sobol’



Wolfgang Mauntz, María Rodríguez Fernández,

Nilay Shah, Costas Pantelides

EPSRC Grant